APPLICATIONS AND METHODS OF GROUP THEORY

IN

ELEMENTARY PARTICLE PHYSICS

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PREFACE

This thesis is divided into two, largely unrelated, sections. The first concerns the production of group theoretic methods which are useful to physicists, whereas the second is an application of group theoretic methods to a specific problem.

In part I (Chapters 1 to 4) I have considered the problem of the evaluation of the Kronecker products of irreducible representations of the compact semisimple groups. The ultimate goal of this study has been the production of simple and efficient methods that can be used to create a single computer package to perform all these operations. The usefulness of this package may be seen from the fact that earlier versions were, to a large extent, instrumental in the derivation and checking of solutions to some of the more difficult problems, such as the $SO_{2k}\downarrow U_k$ branching rule. My contribution in this has been primarily in the development of the SO_{2k}, SO_{2k+1} and Sp_{2k} branching rule method product formulae, the $SO_{2k}\downarrow U_k$ branching rule, and also in the development of the computer package.

In part II (Chapter 5) I have considered an application of tensor operator techniques in the study of multiquark hadrons. This study is an extension of the earlier work of Dr Paul Bickerstaff (1980) whose thesis concerned the derivation of the methodology and also applications to the $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 multiquark configurations. My contribution to this work has been in the calculation of various parameters of the $q^3\bar{q}^3$ configuration. While this configuration is interesting because of its apparent link with baryon-antibaryon production experiments, the magnitude of the problem has in the past been such that a serious study has not been seriously attempted. The Bickerstaff methodology alleviates this problem to a large extent.

Much of this thesis has been published or submitted for publication. A paper, written with Dr R.C. King and Professor B.G. Wybourne, containing many of the results of Part I has been submitted to the Journal of Physics (Black et al. 1982). The results in part II have already been published (Black and Wybourne 1981).

I would like to express my thanks to my supervisor, Professor B.G. Wybourne, for his guidance, support and motivation. Also I would like to thank Dr R.C. King for his suggestions and results, which provided much impetus for the work in part I. Thanks are also due to Dr P.H. Butler and my fellow research students Paul Bickerstaff, Ric Haase and Mike Reid for computing assistance and for many useful and stimulating conversations. Ric and his accomplice David also deserve mention for some amusing interludes. I also thank the University Grants Committee for financial support, and Janet Warburton for her typing of this thesis.

To my parents, I would like to express my deep gratitude for their continued support and for making my education possible and enjoyable. Thanks too to Valerie for her perserverance and support.

CONTENTS

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ABSTRACT

The use of Schur function methods in the evaluation of Kronecker products of irreducible representations of the compact semisimple Lie groups is reviewed. For irreducible representations of the classical groups, explicit, unambiguous and rank-independent general formulae are derived. In particular, for the group SO_{2k}, a technique necessitating the derivation of new branching rules is used to obtain the appropriate formulae. The same technique is used to obtain efficient methods for the evaluation of Kronecker products of the irreducible representations of the exceptional groups. An account is given on computer algorithms developed to implement these methods and formulae.

Tensor operator methods are used to evaluate colour-spin matrix elements of the multiquark hadrons $q^3\bar{q}^3$. The static spherical cavity approximation to the MIT bag model is used to calculate the masses of the S-wave $q^3\bar{q}^3$ states. Many of these states are found to have masses below the baryon-antibaryon threshold, but none are found below the triple meson threshold. Dissociation calculations are performed on the states which have nucleon-antinucleon content. Some comments concerning the relationship of the states to experiment are made.

PART I

CHAPTER 1

INTRODUCTION

The theory of Lie groups has become, in recent years, an indispensable tool in theoretical physics. This is especially true in fundamental research fields such as elementary particle and quantum field theory. Vital to this application is the calculation of several group theoretic properties such as dimensions, branching rules, indices and products of irreducible representations (irreps) of groups, since knowledge of these properties is a prerequisite to quantitative calculations. It is therefore important to the physicist that there exist either tables of these properties or efficient and fast methods for calculating them. Part I of this thesis is primarily concerned with the Kronecker products of irreps of compact semisimple Lie groups. In particular the object is the production of simple, efficient and unambiguous methods for evaluating these products, with a view to computer implementation.

Historically, the study of Lie groups has been approached from two directions. The first, introduced by Lie (Lie and Scheffers, 1893) and Cartan (1894), is associated with infinitesimal operators and weights. It is familiar to physicists because of its link with the Hilbert space techniques of quantum mechanics. The second approach, due originally to Frobenius (Serre 1968) and

Schur (1901), was greatly developed by Littlewood (1940). It is concerned with the characters of groups of matricies, and as such has an affinity to standard tensor and spinor methods. In general the two approaches provide complementary insights into the uses and problems of groups. Both approaches lead to methods for evaluation of Kronecker products, but not equally successful ones. The character methods have the virtue of producing rank independent general formulae for the classical semisimple Lie groups. The weight space techniques, in contrast, do not produce general formulae or rank independent algorithms, and are somewhat tedious, even for low rank groups. Nevertheless these methods may be usefully employed, especially in computer algorithms, to calculate Kronecker products for both classical and exceptional Lie groups (Englefield 1981, Slansky 1981). However the object of this thesis is best served by concentrating on the character approach, using weight space results as required.

The superiority of the character methods depends almost entirely on the powerful algebra of Schur functions. This leads directly to the Kronecker product rule for the covariant irreps of the unitary group, which was first produced by Littlewood and Richardson (1934). Schur function manipulation leads to analogous formulae for the tensor irreps of the orthogonal and symplectic groups (Newell 1951, Littlewood 1958). Methods to evaluate Kronecker products involving spinor irreps of the orthogonal groups were first produced by Butler and Wybourne (1969), but the first general formulae for these products are due to King (1975a). Generalisations in notation using composite Young diagrams (Abramsky and King 1970, King 1970, cf Littlewood 1944 p398) have led to a simple formula for the product of mixed tensor irreps of the unitary group (King 1971).

However Schur function methods have, in the past, failed to produce simple Kronecker product formulae for the even dimensional notation group, SO_{2k}, (King et al. 1981). This has been a major gap in the results for the classical Lie groups. In the exceptional groups the situation is somewhat worse as no formulae at all may be obtained, although Schur function methods are useful in the algorithms that exist (Wybourne and Bowick 1977). To help alleviate this problem, King (1981a) derived a new method of performing products from the weight space method of Racah (1964) and Speiser (1964). This method still does not produce general formulae for the exceptional group products, but it is a considerable improvement on previous methods. The method may also be applied to the classical Lie groups, including SO_{2k}, to produce general product formulae (section 3.4 and 3.5). This application was motivated by recent work of Girardi, Sciarrino and Sorba (1981, 1982) were generalised Young diagram methods for evaluating Kronecker products of classical Lie groups are produced from weight space arguments.

In the next chapter the important problem of the labelling of irreps of the compact semisimple Lie groups is considered. The labelling adopted makes explicit the relationship between representations and Schur functions. The elementary properties of Schur functions and Schur function series are also reviewed. Always necessary when Schur functions are applied in a group context are modification rules to ensure only standard labels are produced. These are listed in full.

In chapter three the Kronecker product problem is considered in depth. Using King's branching rule method new rules are derived for the classical Lie groups SO_{2k+1}, SO_{2k} and Sp_{2k}. A key step in this is the derivation of the difficult branching rule for $SO_{2k} \downarrow U_k$. Several examples are given of the use of the formulae.

Chapter four considers the computer implementation of the products and other group properties. The formulae given in chapter three are both unambiguous and extremely efficient and are therefore ideally suited to such implementation. A program, called SCHUR, has been written to perform both Schur function and group operations. General principles and examples of the use of SCHUR are given.

CHAPTER 2

SCHUR FUNCTIONS AND REPRESENTATIONS OF LIE GROUPS: LABELLING AND BASIC CONCEPTS

In the labelling of both Schur functions and representations of compact semisimple Lie groups, partitions of integers have an important role to play. A partition of the positive integer ℓ and p parts $\lambda_1, \lambda_2, \cdots \lambda_p$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_p = \ell$ is denoted by $(\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_p)$ or often just as (λ). The integer ℓ is often termed the weight of the partition. Any such partition (λ) may be used to specify a regular Young diagram which is made up of ℓ boxes arranged in p rows. In this the i-th row has λ_i boxes as in figure (2.1a). The partition $(\tilde{\lambda})$, conjugate to (λ) may be obtained from the Young diagram by reflecting the diagram about the main diagonal (figure 2.1b). Thus $\tilde{\lambda}_j$ is given by the length of j-th column of the (λ) diagram. An alternate notation for partitions is given by Frobenius (Littlewood 1940 section 5.1). The numbers a_i are defined as the number of boxes

Figure 2.1 (a) Young diagram (b)Conjugate diagram

(31) $(211) = (\widetilde{31})$

in the i-th row of a diagram, (λ), to the right of the main diagonal, that is $\lambda_i - i$. The numbers b_j are defined as the number of boxes beneath the main diagonal in the j-th column of the diagram, that is $b_j = \tilde{\lambda}_j - j$. The partition is then specified by

$$(\lambda) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where r is the Frobenius rank of the partition. Clearly the conjugate partition is obtained by interchanging the a_i and b_i

(2.1) SCHUR FUNCTIONS

A Schur function (S-function) is a particular type of symmetric function defined on a set of indeterminates. Each S-function is defined so as to correspond to an irreducible representation (irrep) of the symmetric group, S_n. Whilst the actual combinatoric structure is not relevant to this thesis, detailed accounts exist in the literature (Littlewood 1940, Wybourne 1970, Macdonald, 1979). Schur functions, like all symmetric functions, may be labelled by partitions. Partitions labelling S-functions are denoted by $\{\lambda\}$, following Littlewood.

Many operations may be defined on S-functions, but only two of these are of particular interest here: the outer product of S-functions and S-function division. The product of two S-functions defined on different sets of indeterminates is called the outer product. The product S-function obtained may be decomposed into a sum of S-functions. This can be written as

$$\{\lambda\}\cdot\{\mu\} \equiv \{\lambda\cdot\mu\} = \sum_{\nu} m^{\nu}_{\lambda\mu} \ \{\nu\} \tag{2.1}$$

where $m^{\nu}_{\lambda\mu}$ is the multiplicity of $\{\nu\}$ in the product. Actual evaluation of outer products is performed using the well known Littlewood-Richardson rule (Littlewood and Richardson 1934, Littlewood 1940, p94). Extensive tabulations of S-function outer products are available in the literature, notably Butler's (Wybourne 1970).

Closely related to the operation of outer product is the S-function division (Littlewood 1940 p110). This can be written as

$$\{\nu\}/\{\mu\} \equiv \{\nu/\mu\} = \sum_{\lambda} m^{\nu}_{\lambda\mu} \ \{\lambda\} \tag{2.2}$$

where the coefficients $m^{\nu}_{\lambda\mu}$ are the same as in (2.1). This operation defines the skew S-function $\{\nu/\mu\}$. Thus $\{\nu/\mu\}$ is the sum of $m^{\nu}_{\lambda\mu}$ times the S-functions $\{\lambda\}$ whose outer product with $\{\mu\}$ gives the S-function $\{\nu\}$.

Given these operations an algebra of S-functions may be defined such that

$$\{\lambda\cdot\mu\} = \{\mu\cdot\lambda\}$$

$$\{\lambda\cdot\mu\cdot\nu\} = \{(\lambda\cdot\mu)\cdot\nu\} + \{\lambda\cdot(\mu\cdot\nu)\}$$

$$\{\lambda\cdot(\mu + \nu)\} = \{\lambda\cdot\mu\} + \{\lambda\cdot\nu\}$$

$$\{(\lambda\cdot\mu)/\nu\} = \sum_{\alpha} \{\lambda/\alpha\}\cdot\{\mu/(\nu/\alpha)\}$$

$$\{\lambda/(\mu\cdot\nu)\} = \{(\lambda/\mu)/\nu\} = \{(\lambda/\nu)/\mu\} \tag{2.3}$$

$$\{(\lambda+\mu)/\nu\} = \{\lambda/\nu\} + \{\mu/\nu\}$$

$$\{\lambda/(\mu+\nu)\} = \{\lambda/\mu\} + \{\lambda/\nu\}$$

The use of infinite series of S-functions is an important technique in practical calculations. The following series (King 1975b, King et al. 1981) are particularly useful

$$A = \sum_{\alpha} (-1)^{a/2}\{\alpha\} \qquad B = \sum_{\beta} \{\beta\}$$

$$C = \sum_{\alpha} (-1)^{a/2}\{\gamma\} \qquad D = \sum_{\delta} \{\delta\}$$

$$E = \sum_{\varepsilon} (-1)^{(e+r)/2}\{\varepsilon\} \qquad F = \sum_{\zeta} \{\zeta\}$$

$$G = \sum_{\varepsilon} (-1)^{(e-r)/2}\{\varepsilon\} \qquad H = \sum_{\zeta} (-)^{z}\{\zeta\}$$

$$L = \sum_{m} (-1)^{m}\{1^{m}\} \qquad M = \sum_{m} \{m\} \tag{2.4}$$

$$P = \sum_{m} (-1)^{m}\{m\} \qquad Q = \sum_{m} \{1^{m}\}$$

$$V = \sum_{\omega} (-1)^{q}\{\tilde{\omega}\} \qquad W = \sum_{\omega} (-1)^{q}\{\omega\}$$

$$X = \sum_{\omega} \{\tilde{\omega}\} \qquad Y = \sum_{\omega} \{\omega\}$$

where (α) and (ε) are partitions which in Frobenius notation take the form

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+1 & a_2+1 & \cdots & a_r+1 \end{pmatrix}, \quad (\varepsilon) = \begin{pmatrix} e_1 & e_2 & \cdots & e_r \\ e_1 & e_2 & \cdots & e_r \end{pmatrix} \tag{2.5}$$

(γ) is the partition conjugate to (α).

The partition ( ), giving the S-function $\{0\} = 1$, is assumed to be included in the series A,C,E and G. (δ) is a partition into even parts only and (β) is conjugate to (δ). (ζ) is any partition while (ω) is a partition of an even number into at most two parts, the second of which is q.

Noting equations (2.3) the use of capitals for the infinite series leads to the notation given by the examples

$$\{\lambda/A\} = \sum_{\alpha} (-1)^{a/2}\{\lambda/\alpha\}$$

$$\{\lambda/\gamma\cdot A\} = \sum_{\alpha} (-1)^{a/2}\{\lambda/\zeta\cdot\alpha\} \tag{2.6}$$

$$AB = \sum_{\alpha,\beta} (-1)^{a/2}\{\alpha\cdot\beta\}$$

Most of the series of (2.4) occur as mutually inverse pairs so that

$$AB = CD = EF = GH = LM = PQ = VW = \{0\} = 1 \tag{2.7}$$

The following identities may also be shown

$$\begin{aligned} AL &= CP = E & \qquad BM &= DQ = F \\ CM &= AQ = G & \qquad DL &= BP = M \\ MP &= AD = W & \qquad LQ &= BC = V \end{aligned} \tag{2.8}$$

Occasionally it will be convenient to use the notation $\{1^r;\lambda\}$ to represent an S-function whose corresponding Young diagram is formed by adjoining a single column of length r to the left side of the Young diagram of $\{\lambda\}$. For example, $\{1^{10};21\}$ is equivalent to $\{321^8\}$.

Further identities of S-functions and series of S-functions may be found in appendix I.

(2.2) LABELLING OF REPRESENTATIONS

The connection between Schur functions and Lie groups comes through the theory of invariant matrices (Littlewood 1940, Ch 10). Using this theory Schur (1901) and later Littlewood were able to develop a formalism in which the S-functions on n variables are seen to be the characters of representations of the n x n matrix groups. The groups considered in this thesis are the unitary group, U_n, the special unitary group, SU_n, the orthogonal groups, O_{2k} and O_{2k+1}, the special orthogonal groups, SO_{2k} and SO_{2k+1}, the symplectic group, Sp_{2k}, and the five exceptional groups G_2, F_4, E_6, E_7, and E_8.

In the case of the unitary group, U_n, and hence SU_n the representations corresponding to the S-functions are also irreducible. These irreps are thus labelled by the S-functions themselves. In the other classical groups (that is non-exceptional groups) the representations that correspond to the S-functions are in general reducible. This fact is due to the presence of the symmetric and antisymmetric metric tensors for the orthogonal and symplectic groups respectively. However the irreps of O_n and Sp_{2k} may still be labelled by partitions and are written as $[\lambda]$ and $\langle\lambda\rangle$ respectively. The S-function representations may be reduced to a sum of irreps of the appropriate group. The reductions with their inverses are given by

$$O_n \qquad \{\lambda\} \downarrow [\lambda/D] \tag{2.9a}$$

$$[\lambda] \uparrow \{\lambda/C\} \tag{2.9b}$$

$$Sp_{2k} \qquad \{\lambda\} \downarrow \langle\lambda/B\rangle \tag{2.10a}$$

$$\langle\lambda\rangle \downarrow \{\lambda/A\} \tag{2.10b}$$

Not all the irreps of U_n may be labelled by S-functions. Along with these representations, whose bases are covariant tensors, there are representations associated with contravariant tensors, labelled $\{\bar{\mu}\}$, and also representations whose bases are mixed tensors, labelled by $\{\bar{\mu};\lambda\}$. In this notation the λ partition is associated with the ℓ covariant indices of the basis tensor while the barred μ partition is associated with the m contravariant indices. It is convenient to write $\{\bar{0};\lambda\} = \{\lambda\}$, $\{\bar{\mu};0\} = \{\bar{\mu}\}$. The mixed tensor rotation can be associated with a composite Young diagram formed by placing the Young diagrams for (μ) and (λ) back to back (figure 2.2a). Another sort of composite Young diagram can be formed using the natural labels discussed at the end of this section. In U_n, each mixed tensor label can be written as a single n-part generalised partition (the condition $\lambda_i \geqslant 0$ is dropped) such that

$$\{\bar{\mu};\lambda\} \equiv \{\lambda_1, \lambda_2, \cdots, -\mu_2, -\mu_1\} . \qquad (2.11)$$

The corresponding n-dependent generalised Young diagram is shown in figure (2.2b).

Figure 2.2 Generalised Young diagram

(a) Composite Young diagram (b) n-dependent 'natural' form

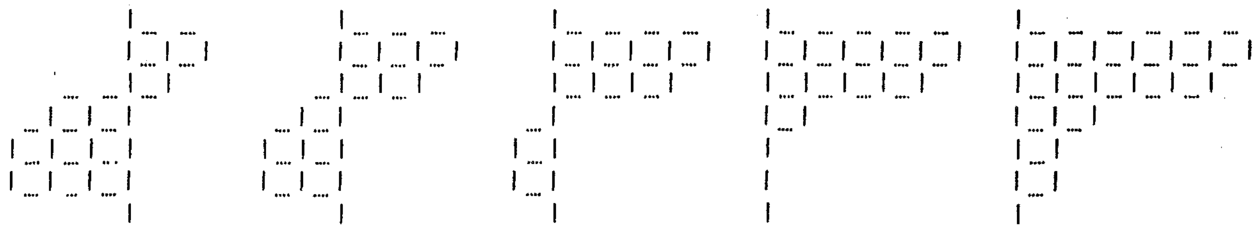
$\{\overline{211};32\}$ $\{\overline{211};32\}$ in U(6)

In the orthogonal groups, as well as the tensor irreps given by $[\lambda]$, there exist double-valued or spinor representations (Brauer and Weyl 1935) which are denoted by $[\Delta;\lambda]$. In this notation Δ is the fundamental spinor representation of dimension 2^k for O_{2k} and O_{2k+1}. $[\Delta;\lambda]$ is the principal representation that appears in the Kronecker product of Δ with $[\lambda]$. (Littlewood 1940, p256).

For each of the linear groups there exists a one dimensional irrep ε which maps each group element onto its determinant. The group elements of the unimodular groups SU_n, SO_n and Sp_n all have determinant +1 by definition, and hence the appropriate irreps ε for these groups are {0}, [0] and <0> respectively. However for U_n and O_n this is not the case. For U_n, ε is the irrep $\{1^n\}$ with an inverse $\varepsilon^{-1} = \bar{\varepsilon} = \{\overline{1^n}\}$. For O_n all group elements have determinant +1 or -1 hence $\varepsilon = \varepsilon^{-1}$ and $\varepsilon \times \varepsilon = [0]$.

The product of the irrep ε with another irrep is also an irrep, and inequivalent irreps related by some power of ε are said to be associated. For U_n there are an infinite number of irreps associated with any given irrep, one of which will be labelled by a partition with less than n parts. For instance (see figure 2.3)

Figure 2.3 Associated irreps of U(5).



$\{\overline{332};21\}$ $\{\overline{221};32\}$ $\{\overline{11};43\}$ $\{541\}$ $\{65211\}$

521 4 3

... $\{\overline{6^2 4}1\}$, $\{\overline{5^2 3}1\}$, $\{\overline{4^2 2};1\}$, $\{\overline{3^2 2};21\}$, $\{\overline{2^2 1};32\}$, $\{\overline{1^2};43\}$, $\{541\}$, $\{6521^2\}$ ... are all associated irreps of U_5. They are related by the addition of +1 to each part of the label given by (2.11). In particular the covariant irrep of less than n parts is related to the general mixed tensor irrep by

$$\{\bar{\mu};\lambda\} = \bar{\varepsilon}^{\mu_1}\{\mu_1+\lambda_1,\mu_1+\lambda_2,\ \cdots\ ,\mu_1-\mu_3,\mu_1-\mu_2,0\} \qquad (2.12)$$

The fact that, under the restriction of U_n to SU_n, all associated irreps become equivalent (since ε=1) allows all irreps of SU_n to be denoted by single partitions of less than n parts. In general the Kronecker product of any real power of ε with an irrep of U_n is an irrep of U_n. If the power of ε is non-integer the irrep is not a true irrep of U_n since the representation is multivalued. Particularly of interest is the case of half integer powers of ε, as these correspond to double-valued representations analogous to the spinor representations of O_n.

For O_n there can only be one inequivalent associate of any irrep and often there is none. Irreps for which the character is zero for all group elements having determinant -1 will not possess an inequivalent associate. Such an irrep is termed self-associate. Following Murnaghan (1938, p276) the irreps associated with $[\lambda]$ and $[\Delta;\lambda]$ are denoted $[\lambda]^* = \varepsilon[\lambda]$ and $[\Delta;\lambda]^* = \varepsilon[\Delta;\lambda]$ respectively. While for O_{2k+1} no irrep is self-associate, for O_{2k} all the spinor irreps and the tensor irreps labelled by partitions with k parts are self-associate.

Under the restriction $O_n \downarrow SO_n$, ε becomes [0] and so the distinction between an irrep and its associate is lost. This means that the asterisk labels are not required for SO_n. However under this restriction only those irreps which are not self-associate remain irreducible. The self-associate irreps of O_{2k} on restriction to SO_{2k} reduce into two inequivalent irreps of equal dimension. This process may be described by (King et al. 1981)

$$O_{2k} \downarrow SO_{2k} \qquad [\lambda] \downarrow [\lambda]_+ + [\lambda]_- \tag{2.13a}$$

$$[\Delta;\lambda] \downarrow [\Delta;\lambda]_+ + [\Delta;\lambda]_- \tag{2.13b}$$

where in the first case (λ) is understood to be of k parts. In SO_{2k} these irreps are related by an involutory outer automorphism, $\dagger$, such that

$$[\lambda]^\dagger = [\lambda] \qquad p < k \tag{2.14a}$$

$$[\lambda]^\dagger_\pm = [\lambda]_\mp \qquad p = k$$

$$[\Delta;\lambda]^\dagger_\pm = [\Delta;\lambda]_\mp \qquad p \leqslant k \quad . \tag{2.14c}$$

The fact that there exist equivalences within the notation means that a subset of all the possible labels can be defined as standard. These are specified in table (2.1) for the classical Lie groups U_n, SU_n, O_n, SO_n and Sp_{2k}.

The irreps of the exceptional groups G_2, F_4, E_6, E_7 and E_8 may be usefully labelled by employing the labelling of a maximal classical Lie subgroup (Wybourne and Bowick

1977, King and Al-Qubanchi 1981a). There are many choices of maximal subgroup available. In this thesis the criterion of most importance is the amount to which Kronecker products are facilitated. The choices are SU_3, SO_9, the product group $SU_2 \times SU_6$, SU_8 and SU_9 for G_2, F_4, E_6, E_7 and E_8 respectively. Additional constraints are put on these labels so that the exceptional group standard labels are a subset of those of the classical subgroup. Table (2.2) gives a complete specification of the standard labels of the exceptional groups.

The labels described may be used to specify any irrep of the compact semi-simple Lie groups. Several alternative labelling schemes exist and may be useful in many situations. Very closely related to the standard labels are the natural labels of King and Al-Quabanchi (1981a) which are trivially derived from the highest weight vector of the irrep. The natural labels, denoted $\underset{\sim}{\lambda}$, are k dimensional vectors where k is the rank of the Lie algebra corresponding to the Lie group. They are generalised partition labels in the sense that $\underset{\sim}{\lambda} = (\underset{\sim}{\lambda}_1, \underset{\sim}{\lambda}_2, \ldots , \underset{\sim}{\lambda}_k)$ is ordered such that $\underset{\sim}{\lambda}_1 \geqslant \underset{\sim}{\lambda}_2 \geqslant \cdots \geqslant \underset{\sim}{\lambda}_k$, though the $\underset{\sim}{\lambda}_i$ may be negative and half-integer. The relations between the standard labels and the natural labels are given by:

$$(\lambda) = (\lambda_1, \lambda_2, \ldots , \lambda_p, 0, 0, \ldots 0) \tag{2.15a}$$

$$(\bar{\mu};\lambda) = (\lambda_1, \lambda_2, \ldots , \lambda_p, 0, \ldots , 0, -\mu_q, \ldots , -\mu_2, -\mu_1) \tag{2.15b}$$

$$(\Delta;\lambda) = (\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2}, \cdots, \lambda_p+\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots \tfrac{1}{2}) \tag{2.15c}$$

$$(s:\lambda) = (s,\lambda_1,\lambda_2, \ldots, \lambda_p, 0, 0, \ldots 0) . \tag{2.15d}$$

The label $(s:\lambda)$ is used to label the exceptional group E_6 through the group $SU_2 \times SU_6$. In addition to these symbols it is useful to introduce the following notation

$$(\square;\lambda) = (\lambda_1+1, \lambda_2+1, \ldots, \lambda_p+1, 1, 1, \ldots 1) \tag{2.15e}$$

$$(\Delta;\bar{\mu};\lambda) = (\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2}, \ldots, -\mu_2+\tfrac{1}{2}, -\mu_1+\tfrac{1}{2}) \tag{2.15f}$$

The '±' notation, useful for SO_{2k} is described in natural labels by

$$\underset{\sim}{\lambda}_{\pm} = (\lambda_1,\lambda_2, \ldots, \lambda_{k-1}, \pm\lambda_k)_+ = (\underset{\sim}{\lambda}_1,\underset{\sim}{\lambda}_2, \ldots, \underset{\sim}{\lambda}_{k+1}, \mp\underset{\sim}{\lambda}_k)_- \tag{2.16}$$

The irreps of the Lie groups are commonly labelled using Dynkin's (1957) scheme. The relation between the Dynkin labels and the natural labels, which is quite trivial, is given by King and Al-Qubanchi (1981a). Often properties of irreps, such as dimensions and second order Casimir invariants are used for labelling purposes. The formulae for the dimensions of the classical groups using partition labelling are well known (Wybourne 1970, El Samra and King, 1979), however the general formulae for dimensions of the exceptional groups and second order Casimir invariants for both the classical and exceptional Lie groups must be derived from the original work by Dynkin (1957) or from

Wybourne (1974). The formulae for the second order Casimir invariants of the classical groups are given in table (2.3) and the formulae for the dimensions and second order Casimir invariants of the exceptional groups in table (2.4).

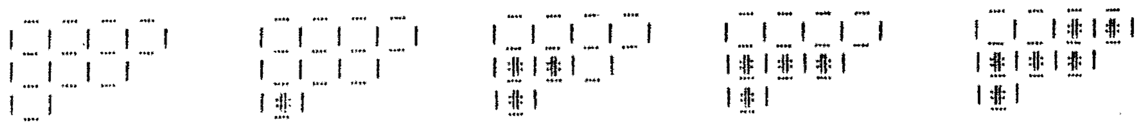
(2.3) MODIFICATION RULES

While the standard labels of tables (2.1) and (2.2) suffice to completely label the irreps of both the classical and exceptional Lie groups, in many calculations non-standard labels will arise. In a particular group, G, the character corresponding to a non-G-standard label is found to either vanish or be equal to the character of a G-standard irrep or the negative of such a character. The equivalence relations between non-standard and standard labels are known as modification rules. Some trivial modification rules are given in tabel (2.5).

There are two main types of modification rules to be considered. The first type,only necessary for the classical groups, are used when the number of parts in a partition exceeds the allowed number, k. The second type are often necessary after the employment of weight space techniques or related methods (such as in section 3.2) where the condition on the labels (λ) that $\lambda_i \geqslant 0$ or the auxiliary conditions in table (2.2) may be broken.

Early work on the first type of rule was done by Murnaghan (1938, p282) and Newell (1951). The most useful expression of these rules is made in terms of boundary hook removal (King 1971, King 1975b, King et al. 1981). Boundary hook removal is the process where a continuous run of boxes is removed from the right boundary of a Young diagram starting at the foot of the first column. The notation (λ-h) is used for a diagram (λ) modified by the removal of a boundary hook of length h. If the diagram (λ-h) is not regular, that is, if the columns of (λ-h) are not weakly decreasing from left to right the character of the irrep corresponding to the label will vanish. Figure (2.4) gives several examples of hook length removal. Table (2.6) gives a complete list of the modification rules of this type.

Figure 2.4 Hook length removal



	h=1	h=3	h=4	h=6
(431)	(431-h)	(431-h)	(431-h)	(431-h)
	=(43)	vanishes	=(4)	=(2)

The modification rules of the second type may be derived from the Weyl reflection symmetry of the weights of an irrep of a group (King and Al-Qubanchi 1981b). The key rule of this type (Murnaghan 1937) is

$$(\lambda_1, \lambda_2, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_k)$$

$$= - (\lambda_1, \lambda_2, \ldots, \lambda_{i+1}-1, \lambda_i+1, \ldots, \lambda_k). \qquad (2.17)$$

This rule is valid if $i = 1,2, \ldots , k-1$ for the classical groups of rank k and for $i = 2,3, \ldots , k-1$ for the exceptional groups of rank k. Repeated use of this rule allows any sequence of $\underset{\sim}{\lambda}_j$s to be ordered such that $\underset{\sim}{\lambda}_1 \geqslant \underset{\sim}{\lambda}_2 \geqslant \ldots \geqslant \underset{\sim}{\lambda}_k$. In addition to this rule there are several rules special to particular groups. King and Al-Qubanchi (1981a) have evaluated these and the complete set of their rules using natural labels is given in table (2.7).

The need will arise, later in the thesis, to modify generalised partitions to standard ones for the groups SO_{2k}, SO_{2k+1} and Sp_{2k}. To achieve this, the rules of table (2.7) must be applied in a purposeful sequence to eliminate all negative parts of $\underset{\sim}{\lambda}$ (i.e. $(\bar{\mu};\lambda)$ or $(\Delta;\bar{\mu};\lambda)$).

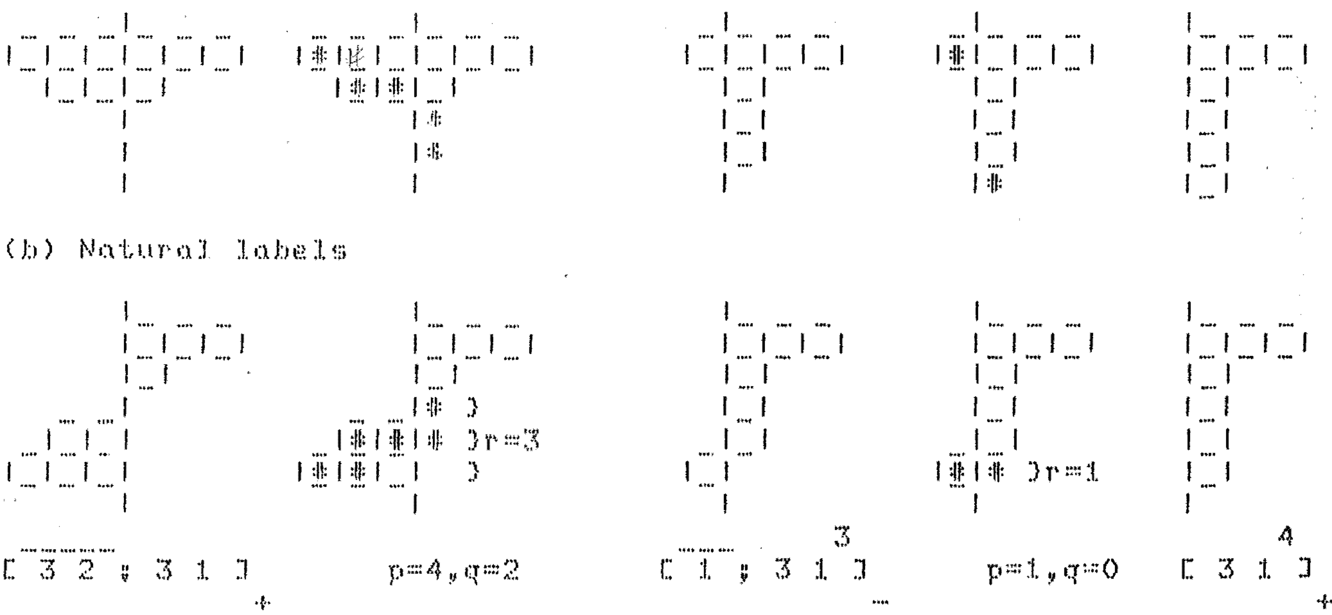
For SO_{2k} the method is to first apply equation (2.16) to change $\underset{\sim}{\lambda}_k$ (that is either $-\mu_1$ or $-\mu_1+\frac{1}{2}$) to $-\underset{\sim}{\lambda}_k$ and change the sign of the '±' index on $\underset{\sim}{\lambda}_\pm$. This process is the same as adding $2|\underset{\sim}{\lambda}_k|$ to $\underset{\sim}{\lambda}_k$ or equivalently adding $2\mu_1$ for the case of $(\bar{\mu};\lambda)$ and $2\mu_1-1$ for the case $(\Delta;\bar{\mu};\lambda)$. This process can be summarised as the adding of a continuous boundary hook of length $2|\underset{\sim}{\lambda}_k|$ to the generalised partition $\underset{\sim}{\lambda}$, and the multiplication by a sign factor determined by the number of uses of (2.17). Similar rules can be derived for SO_{2k+1} and Sp_{2k} using the special modification rules of table (2.7) instead of (2.17). For SO_{2k+1} and Sp_{2k} the lengths of boundary hook to be added on are $2|\underset{\sim}{\lambda}_k| - 1$ and $2|\underset{\sim}{\lambda}_k| - 2$ respectively.

Using standard labels this process can be seen as a combination of hook length removal and addition, and so involves the production of labels $(\overline{\mu-p};\lambda+q)$ and $(\Delta;\mu-p;\lambda+q)$ from $(\bar{\mu};\lambda)$ and $(\Delta;\bar{\mu};\lambda)$, where p and q are boundary hooks. These rules are summarised in table (2.8).

The rules may have to be used repeatedly to produce a standard irrep label. In fact the number of times the rule must be used is just the Frobenius rank of (μ). As an example of these rules consider the modification of the irrep $[\overline{32};31]_+$ in the group SO_{10}. This is illustrated in figure (2.5). The rank k = 5, $\mu_1 = 3$ and $\tilde{\mu}_1 = 2$ so that $p = 4$, $q = 2$, $k - \tilde{\mu}_1+1 = 4$ and hence $r = 3$. This leads to the label $[\bar{1};31^3]_-$ which still requires modification. A second application of the rule leads to the label $[31^4]_+$ with p=1, q=1 and r=5. Alternatively, use of (2.15) produces the same label.

Figure 2.5 SO(2k) modification of $[\overline{32};31]_+$

(a) Composite partition labels



The symbol $\{1^r;\lambda\}$, introduced in section (2.1), may also require modification under some circumstances. If r is less than p, the number of parts of λ, the corresponding Young diagram is irregular. The modification rule is given by (King 1971)

$$\{1^r;\lambda\} = (-1)^c\{1^{p-1};\lambda-h\}, h = p-r-1 \tag{2.18}$$

where all conventions are as in table (2.6).

Table (2.1): Standard labels of the classical groups.

Group	Label	Constraint
U_n	$\{\bar{\mu};\lambda\}$	$p+q \leqslant n$
SU_n	$\{\lambda\}$	$p \leqslant n-1$
O_{2k+1}	$[\lambda],[\lambda]^*$	$p \leqslant k$
	$[\Delta;\lambda],[\Delta;\lambda]^*$	$p \leqslant k$
SO_{2k+1}	$[\lambda]$	$p \leqslant k$
	$[\Delta;\lambda]$	$p \leqslant k$
O_{2k}	$[\lambda]$	$p \leqslant k$
	$[\lambda]^*$	$p \leqslant k-1$
	$[\Delta;\lambda]$	$p \leqslant k$
SO_{2k}	$[\lambda]$	$p \leqslant k-1$
	$[\lambda]_+,[\lambda]_-$	$p = k$
	$[\Delta;\lambda]_+,[\Delta;\lambda]_-$	$p \leqslant k$
Sp_{2k}	$\langle\lambda\rangle$	$p \leqslant k$

(λ) a partition into p parts

(μ) a partition into q parts

Table (2.2): Standard labels of the exceptional groups

Group	Maximal Subgroup	Label		Constant
G_2	SU_3	(λ)	$p \leqslant 2$	$\lambda_1 \geqslant 2\lambda_2$
F_4	SO_9	(λ)	$p \leqslant 4$	$\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4$
		$(\Delta;\lambda)$	$p \leqslant 4$	$\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4$
E_6	$SU_2 \times SU_6$	$(s:\lambda)$	$p \leqslant 5$	$s \geqslant \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5$
E_7	SU_8	(λ)	$p \leqslant 7$	$\lambda_1 \geqslant \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7$
E_8	SU_9	(λ)	$p \leqslant 8$	$\lambda_1 \geqslant 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8$

Table (2.3): Second order Casimir invariants for the classical groups

SU_n: $$C_2\{\lambda\} = \frac{1}{2n}\left[\sum_{i=1}^{n} \lambda_i(\lambda_i - 2i + n + 1) - \frac{\omega_\lambda^2}{n}\right]$$

SO_{2k+1}: $$C_2[\lambda] = \frac{1}{2(2k+1)}\sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i + 1)$$

$$C_2[\Delta;\lambda] = \frac{1}{2(2k+1)}\left[\sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i + 2) + \frac{k}{4}(2k+1)\right]$$

Sp_{2k}: $$C_2\langle\lambda\rangle = \frac{1}{4(k+1)}\sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i + 2)$$

SO_{2k}: $$C_2[\lambda]_{(\pm)} = \frac{1}{4k-1}\sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i)$$

$$C_2[\Delta;\lambda]_{\pm} = \frac{1}{4k-1}\left[\sum_{i=1}^{k} \lambda_i(\lambda_i + 2k - 2i + 1) + \frac{k}{4}(2k-1)\right]$$

where ω_λ is the weight of the label (λ).

Table (2.4): Dimensions and second order Casimir invariants for the exceptional groups.

G_2: $$D(\lambda) = \frac{1}{120}(\lambda_1+\lambda_2+4)(\lambda_1-2\lambda_2+1)(2\lambda_1-\lambda_2+5)(\lambda_1+3)(\lambda_2+1)(\lambda_1-\lambda_2+2)$$

$$C_2(\lambda) = \frac{1}{12}[(\lambda_1-\lambda_2)^2 + \lambda_1\lambda_2 + 5\lambda_1 - \lambda_2]$$

F_4: $$D(\lambda) = \left(\frac{11+2\lambda_1}{11}\right)\left(\frac{5+2\lambda_2}{5}\right)\left(\frac{3+2\lambda_3}{3}\right)\left(\frac{1+2\lambda_4}{1}\right)\left(\frac{8+\lambda_1+\lambda_2}{8}\right)\left(\frac{7+\lambda_1+\lambda_3}{7}\right)$$
$$\left(\frac{6+\lambda_1+\lambda_4}{6}\right)\left(\frac{4+\lambda_2+\lambda_3}{4}\right)\left(\frac{3+\lambda_2+\lambda_4}{3}\right)\left(\frac{2+\lambda_3+\lambda_4}{2}\right)\left(\frac{3+\lambda_1-\lambda_2}{3}\right)$$
$$\left(\frac{4+\lambda_1-\lambda_3}{4}\right)\left(\frac{5+\lambda_1-\lambda_4}{5}\right)\left(\frac{1+\lambda_2-\lambda_3}{1}\right)\left(\frac{2+\lambda_2-\lambda_4}{2}\right)\left(\frac{1+\lambda_3-\lambda_4}{1}\right)$$
$$\left(\frac{10+\lambda_1+\lambda_2+\lambda_3+\lambda_4}{10}\right)\left(\frac{9+\lambda_1+\lambda_2+\lambda_3-\lambda_4}{9}\right)\left(\frac{7+\lambda_1+\lambda_2-\lambda_3+\lambda_4}{7}\right)$$
$$\left(\frac{6+\lambda_1+\lambda_2-\lambda_3-\lambda_4}{6}\right)\left(\frac{5+\lambda_1-\lambda_2+\lambda_3+\lambda_4}{5}\right)\left(\frac{4+\lambda_1-\lambda_2+\lambda_3-\lambda_4}{4}\right)$$
$$\left(\frac{2+\lambda_1-\lambda_2-\lambda_3+\lambda_4}{2}\right)\left(\frac{1+\lambda_1-\lambda_2-\lambda_3-\lambda_4}{1}\right)$$

$$D(\Delta;\lambda) = \left(\frac{12+2\lambda_1}{11}\right)\left(\frac{6+2\lambda_2}{5}\right)\left(\frac{4+2\lambda_3}{3}\right)\left(\frac{2+2\lambda_4}{1}\right)\left(\frac{9+\lambda_1+\lambda_2}{8}\right)\left(\frac{8+\lambda_1+\lambda_3}{7}\right)$$
$$\left(\frac{7+\lambda_1+\lambda_4}{6}\right)\left(\frac{5+\lambda_2+\lambda_3}{4}\right)\left(\frac{4+\lambda_2+\lambda_4}{3}\right)\left(\frac{3+\lambda_3-\lambda_4}{2}\right)\left(\frac{3+\lambda_1-\lambda_2}{3}\right)$$
$$\left(\frac{4+\lambda_1-\lambda_3}{4}\right)\left(\frac{5+\lambda_1-\lambda_4}{5}\right)\left(\frac{1+\lambda_2-\lambda_3}{1}\right)\left(\frac{2+\lambda_2-\lambda_4}{2}\right)\left(\frac{1+\lambda_3-\lambda_4}{1}\right)$$
$$\left(\frac{12+\lambda_1+\lambda_2+\lambda_3+\lambda_4}{10}\right)\left(\frac{10+\lambda_1+\lambda_2+\lambda_3-\lambda_4}{9}\right)\left(\frac{8+\lambda_1+\lambda_2-\lambda_3+\lambda_4}{7}\right).$$

Table (2.4) Continued

$$\left(\frac{6+\lambda_1+\lambda_2-\lambda_3-\lambda_4}{6}\right)\left(\frac{6+\lambda_1-\lambda_2+\lambda_3+\lambda_4}{5}\right)\left(\frac{4+\lambda_1-\lambda_2+\lambda_3-\lambda_4}{4}\right)$$

$$\left(\frac{2+\lambda_1-\lambda_2-\lambda_3+\lambda_4}{2}\right)\left(\frac{\lambda_1-\lambda_2-\lambda_3-\lambda_4}{1}\right)$$

$$C_2(\lambda) = \frac{1}{18}[\lambda_1(\lambda_1+11) + \lambda_2(\lambda_2+5) + \lambda_3(\lambda_3+3) + \lambda_4(\lambda_4+1)]$$

$$C_2(\Delta;\lambda) = \frac{1}{18}[\lambda_1(\lambda_1 + 12) + \lambda_2(\lambda_2+6) + \lambda_3(\lambda_3+4)$$

$$+ \lambda_4(\lambda_4+2) + 11]$$

2 p

E_6: $$D(s:\lambda) = \left(\frac{s+11}{11}\right)\left[\prod_{p=1}^{5}\prod_{q=p+1}^{6}\left(\frac{\lambda_p-\lambda_q+p-q}{p-q}\right)\right]$$

? p 32

$$\left[\prod_{p=1}^{4}\prod_{q=p+1}^{5}\prod_{r=q+1}^{6}\left(\frac{2\lambda_p+2\lambda_q+2\lambda_r+s-\omega_\lambda+44-2(p+q+r)}{44-2(p+q+r)}\right)\right]$$

32

$$C_2(s:\lambda) = \frac{1}{48}\left[s(s+22) + 2\left(\sum_{i=1}^{5}\lambda_i(\lambda_i-2i+7) - \frac{\omega_\lambda^2}{6}\right)\right]$$

(where $\lambda_6 = 0$)

E_7: $$D(\lambda) = \left[\prod_{p=2}^{8}\left(\frac{\lambda_1-\lambda_p+9+p}{9+p}\right)\right]\left[\prod_{p=2}^{7}\prod_{q=p+1}^{8}\left(\frac{\lambda_p-\lambda_q+q-p}{q-p}\right)\right]$$

$$\left[\prod_{p=2}^{5}\prod_{q=p+1}^{6}\prod_{r=q+1}^{7}\prod_{s=r+1}^{8}\left(\frac{\frac{\omega_\lambda}{2}-\lambda_p-\lambda_q-\lambda_r-\lambda_s+p+q+r+s-13}{p+q+r+s-13}\right)\right]$$

$$C(\lambda) = \frac{1}{288}\ [7\sum_{i=1}^{7}\lambda_i^2 - 2\sum_{i=1}^{6}\sum_{j=i+1}^{7}\lambda_i\lambda_j$$

$$+ 4(49\lambda_1 + 5\lambda_2 + \lambda_3 - 3\lambda_4 - 7\lambda_5 - 11\lambda_6 - 15\lambda_7)]$$

(where $\lambda_8 = 0$)

Table (2.4) Continued

E_8: $$D(\lambda) = \left[\prod_{p=2}^{9}\left(\frac{\lambda_1-\lambda_p+20+p}{20+p}\right)\right]$$

$$\left[\prod_{p=2}^{8}\prod_{q=p+1}^{9}\left(\frac{\lambda_p-\lambda_q+q-p}{q-p}\right)\left(\frac{\lambda_1+\lambda_p+\lambda_q-\frac{\omega_\lambda}{3}+28-p-q}{28-p-q}\right)\right]$$

$$\left[\prod_{p=2}^{7}\prod_{q=p+1}^{8}\prod_{r=q+1}^{9}\left(\frac{\frac{\omega_\lambda}{3}-\lambda_p-\lambda_q-\lambda_r+p+q+r-10}{p+q+r-10}\right)\right]$$

$$C_2(\lambda) = \frac{1}{270}\,[4\sum_{i=1}^{8}\lambda_i^2 - \sum_{i=1}^{7}\sum_{j=i+1}^{8}\lambda_i\lambda_j$$

$$+ 3(68\lambda_1 + 2\lambda_2 - \lambda_3 - 4\lambda_4 - 7\lambda_5 - 10\lambda_6$$

$$- 13\lambda_7 - 16\lambda_8)]$$

(where $\lambda_9 = 0$)

ω_λ is the weight of the partition (λ).

Table (2.5) Modification rules (1)

Group	Rule	Condition
SU_n	$\{\lambda\} = \{\bar{\lambda}\} = 0$	$p > n$
	$\{\lambda\} = \{\lambda/n^{\lambda_n}\}$	$p = n$
	$\{\bar{\mu};\lambda\} = \bar{\varepsilon}^{\mu_1}\{\rho\} = \{\rho\}$	$p+q \leqslant n$
O_{2k}	$[\lambda]^* = [\lambda]$	$p=k$
	$[\Delta;\lambda] = [\Delta;\lambda]$	$p \leqslant k$
SO_{2k}	$[\lambda] = [\Box;\lambda/1^k]$	$p = k$
	$[\lambda]_{\pm} = [\Box;\lambda/1^k]_{\pm}$	$p = k$
	$[\Delta;\lambda] = [\Delta;\lambda]_+ + [\Delta;\lambda]_-$	$p \leqslant k$
	$[\Box;\lambda] = [\Box;\lambda]_+ + [\Box;\lambda]_-$	$p = k$

Table (2.6): Modification rules (2).

Group	Rule	Hook length
U_n, SU_n	$\{\bar{\mu};\lambda\} = (-1)^{c+d-1}\{\overline{\mu-h};\lambda-h\}$	$h = p+q-n-1 \geqslant 0$
O_{2k+1}	$[\lambda] = (-1)^{c-1}[\lambda-h]^*$	$h = 2p-2k-1 > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k-1 > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]^*$	$h = 2p-2k-2 \geqslant 0$
	$[\Delta;\lambda]^* = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-2 \geqslant 0$
SO_{2k+1}	$[\lambda] = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k-1 > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-2 \geqslant 0$
O_{2k}	$[\lambda] = (-1)^{c-1}[\lambda-h]^*$	$h = 2p-2k > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k \geqslant 0$
SO_{2k}	$[\lambda] = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-1 \geqslant 0$
	$[\Box;\lambda] = (-1)^{c-1}[\Box;\lambda-h]$	$h = 2p-2k-2 \geqslant 0$
	$[\Delta;\lambda]_{\pm} = (-1)^{c}[\Delta;\lambda-h]_{\mp}$	$h = 2p-2k-1 \geqslant 0$
	$[\Box;\lambda]_{\pm} = (-1)^{c-1}[\Box;\lambda-h]_{\mp}$	$h = 2p-2k-2 \geqslant 0$
Sp_{2k}	$\langle\lambda\rangle = (-1)^{c}\langle\lambda-h\rangle$	$h = 2p-2k-2 \geqslant 0$

(λ) and (μ) are partitions of p and q respectively. c and d are columns of (λ) and (μ) in which the boundary hook ends.

Table (2.7): Weight space modification rules

$$\lambda = -\mu$$

(a) Classical groups of rank k

Group	$-\mu$	
All groups	$-(\lambda_1, \cdots, \lambda_{i+1}-1, \lambda_i+1, \cdots, \lambda_k)$	$i = 1,2, \ldots, k-1$
SU_{k+1}	$-\{\lambda_1-\lambda_k-1, \lambda_2-\lambda_k-1, \ldots, \lambda_{k-1}-\lambda_k-1, -\lambda_k-2\}$	
SO_{2k+1}	$-[\lambda_1,\lambda_2, \ldots, \lambda_{k-1}, -\lambda_k-1]$	
Sp_{2k}	$-\langle\lambda_1,\lambda_2, \ldots, \lambda_{k-1}, -\lambda_k-2\rangle$	
SO_{2k}	$-[\lambda_1,\lambda_2, \ldots, -\lambda_k-1, -\lambda_{k-1}-1]$	

(b) Exceptional groups

Group	Maximal subgroup	$-\mu$	$\varepsilon \geqslant 0$
All groups	-	$-(\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_k)$	$(i = 2,3, \ldots, k-1)$
G_2	SU_3	$-(\lambda_1,\lambda_2-\varepsilon)$	$-\lambda_1 + 2\lambda_2-1$
F_4	SO_9	$-(\lambda_1+\varepsilon,\lambda_2-\varepsilon,\lambda_3-\varepsilon,\lambda_4-\varepsilon)$	$\frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3+\lambda_4-1)$
E_6	$SU_2 x SU_6$	$-(\lambda_1-\lambda_6-1,\lambda_2-\lambda_6-1,\lambda_3-\lambda_6-1,\ldots,$ $-\lambda_5-\lambda_6-1,-\lambda_6-2)$	
		$-(\lambda_1+\varepsilon,\lambda_2-\varepsilon,\lambda_3-\varepsilon,\lambda_4-\varepsilon,\lambda_5,\lambda_6)$	$\frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5-\lambda_6-2)$
E_7	SU_8	$-(\lambda_1,\lambda_2-\varepsilon,\lambda_3-\varepsilon,\lambda_4-\varepsilon,\lambda_5-\varepsilon,$ $\lambda_6,\lambda_7)$	$\frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$ $-\lambda_6-\lambda_7-2)$
E_8	SU_9	$-(\lambda_1,\lambda_2-\varepsilon,\lambda_3-\varepsilon,\lambda_4-\varepsilon,\lambda_5,\lambda_6,$ $\lambda_7,\lambda_8)$	$\frac{1}{3}(-\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4$ $-\lambda_5-\lambda_6-\lambda_7-\lambda_8)$

In addition G_2, F_4, E_7 and E_8 may be modified using the rules appropriate to their classical maximal subgroup (table 2.7a).

Table (2.8): Composite partition modification rules

Group	Rule	hook length
SO_{2k}	$[\bar{\mu};\lambda]_{\pm} = (-)^{r-1}[\overline{\mu-p};\lambda+q]_{\mp}$	$q = \mu_1-\tilde{\mu}_1+1 \geqslant 0$
	$[\Delta;\bar{\mu};\lambda]_{\pm} = (-1)^{r-1}[\Delta;\overline{\mu-p};\lambda+q]_{\mp}$	$q = \mu_1-\tilde{\mu}_1 \geqslant 0$
SO_{2k+1}	$[\bar{\mu};\lambda] = (-1)^{r}[\overline{\mu-p};\lambda+q]$	$q = \mu_1-\tilde{\mu}_1 \geqslant 0$
	$[\Delta;\bar{\mu};\lambda] = (-1)^{r}[\Delta;\overline{\mu-p};\lambda+q]$	$q = \mu_1-\tilde{\mu}_1-1 \geqslant 0$
Sp_{2k}	$\langle\bar{\mu};\lambda\rangle = (-1)^{r}\langle\overline{\mu-p};\lambda+q\rangle$	$q = \mu_1-\tilde{\mu}_1-1 \geqslant 0$

In all cases $p = \mu_1+\tilde{\mu}_1-1$

q begins in the $(k-\tilde{\mu}_1+1)$th row of the Young diagram corresponding to (λ) and finishes in the $(k-r+1)$th row.

CHAPTER 3

KRONECKER PRODUCTS FOR COMPACT SEMISIMPLE LIE GROUPS

This chapter is devoted to the derivation of efficient methods of evaluating the Kronecker products of the compact semisimple Lie groups. In the context of this thesis, efficient evaluation means methods which are fast, simple and which also produce rank independent formulae.

In the next section the application of Schur functions to Kronecker product calculation is considered. Little of this material is new, but it demonstrates both the power and the limitations of the S-function algebra. The power is evident in the production of general formulae which are no harder to evaluate for very high rank groups than for trivially ranked groups. The limitations arise in the inability to produce formulae for the exceptional groups or simple ones for the group SO_{2k}. King (1981a) has developed a new method of Kronecker product evaluation using products of a subgroup. This is discussed and examined in section (3.2). The classical group products causing difficulty, those of SO_{2k}, may be evaluated using this method. First though, the branching rule to a subgroup where products are easily calculated must be found. The new branching rules for $SO_{2k}\downarrow U_k$ are derived in section (3.3), allowing the product formulae to be presented in section (3.4). Formulae for the products of SO_{2k+1} and Sp_{2k} can also be derived by means of the branching rule method. The formulae and their efficiency when compared with those of section (3.1) are considered in section (3.5). Finally in this chapter the application of the branching rule method to products of the exceptional groups is considered in section (3.6).

(3.1) KRONECKER PRODUCTS USING SCHUR FUNCTION TECHNIQUES

The Kronecker product of two representations of a linear group is isomorphic to the multiplication of the corresponding characters. For representations whose characters are S-functions on the eigenvalues of the group elements, this multiplication corresponds to the outer product of those S-functions (Littlewood 1940). This fact has great application in evaluating the Kronecker products of irreps of the classical groups.

Since the characters of the purely covariant and purely contravariant irreps of U_n are just S-functions, Kronecker products of these irreps are just given by the Littlewood-Richardson rule, so that

$$\begin{aligned} U_n \qquad & \{\lambda\} \times \{\mu\} = \{\lambda . \mu\} \\ & \{\overline{\lambda}\} \times \{\overline{\mu}\} = \{\overline{\lambda . \mu}\} \ . \end{aligned} \tag{3.1}$$

For the mixed tensor irreps of U_n the product formula is more complicated due to the additional constraint that the mixed tensors associated with the irreps must be traceless under the contraction of any contravariant index with a covariant index. Abramsky and King (1970, King 1970, 1971) derived a compact formula in which the appropriate symmetrisations and extractions of traces are carried out. This may be written as

$$U_n \qquad \{\bar{\mu};\lambda\} \times \{\bar{\rho};\nu\} = \sum_{\alpha,\beta} \{\overline{(\mu/\alpha)\cdot(\rho/\beta)};(\lambda/\beta)\cdot(\nu/\alpha)\} \ , \qquad (3.2)$$

where $\{\alpha\}$ and $\{\beta\}$ are any S-functions that fit. Equation (3.1) is a special case of this, being produced when either $(\mu) = (\rho) = (0)$ or when $(\lambda) = (\nu) = (0)$. Like (3.1) this formula is an n-independent result, however some non-standard labels may appear in the final result for particular values of n. These terms will require the use of the U_n modification rules found in table (2.6).

An alternate method for computing products, and one which avoids the modification process needed for (3.2), uses the special irrep ε. This may be used to convert a mixed tensor irrep into a product of a power of $\bar{\varepsilon}$ and a purely covariant tensor irrep (using the identity (2.12). These may be multiplied together to form a sum of such products which may be converted back to a sum of mixed tensor irreps.

As an example consider the product $\{\bar{1};1\} \times \{\bar{1};1\}$ in U_3. Formula (3.2) gives the U_n result

$$U_n \qquad \{\bar{1};1\} \times \{\bar{1};1\} = \{\bar{2};2\} + \{\bar{2};1^2\} + \{\bar{1}^2;2\} + \{\bar{1}^2;1^2\}$$
$$+2\{\bar{1};1\} + \{\bar{0};0\} \ .$$

However, in U_3 the label $\{\overline{1^2};1^2\}$ is non-standard as the sum of the number of parts is four. On modification by the appropriate rule in table (2.6) it is found to vanish, and hence the term does not contribute to the product in U_n, n=3. The U_3 result is then just

$$U_3 \qquad \{\bar{1};1\} \times \{\bar{1};1\} = \{\bar{2};2\} + \{\bar{2};1^2\} + \{\overline{1^2};2\} + 2\{\bar{1};1\} + \{0\}.$$

In the alternate method, the first step is to note that for U_3, equation (2.12) gives $\{\bar{1};1\} = \bar{\varepsilon}\{21\}$. Then, using (3.1) followed by the simpler U_n modification rule in table (2.5), and then (2.12)

$$
\begin{aligned}
U_3 \qquad \{\bar{1};1\} \times \{\bar{1};1\} &= \bar{\varepsilon}^2 \times \{21\} \times \{21\} \\
&= \bar{\varepsilon}^2 \times (\{2^2 1^2\} + \{2^3\} + \{31^3\} + 2\{321\} \\
&\quad + \{3^2\} + \{41^2\} + \{42\}) \\
&= \bar{\varepsilon}^2 \times (\{2^3\} + 2\{321\} + \{3^2\} + \{41^2\} \\
&\quad \{42\}) \\
&= \{0\} + 2\{\bar{1};1\} + \{\bar{2};1^2\} + \{\bar{1}^2;2\} + \{\bar{2};2\} \ .
\end{aligned}
$$

The use of (3.2) is, in general, superior to this alternate method as the alternate method only produces results for a particular n. In addition, in high rank groups the identity (2.12) may produce high weight covariant labels from low weight mixed tensor labels. For instance in U_{18}, (2.12) gives that $\{\bar{1};1\} = \bar{\varepsilon}^2\{21^{16}\}$. However the use of ε, and especially half-integral powers of ε in this way, has many advantages which will be exploited in the use of the branching rule method in the Kronecker products of irreps of SO_{2k} and SO_{2k+1}.

For the groups O_n, SO_n and Sp_{2k} the S-function representations are reducible. However use of reductions (2.9) and (2.10) allows the products of the tensor irreps of O_n, SO_{2k+1} and SO_{2k} to be evaluated.

For Sp_{2k} the rule is just (King 1975a)

$$Sp_{2k} \quad \langle\lambda\rangle \times \langle\mu\rangle = \langle((\lambda/A)\cdot(\mu/A)/B\rangle \quad . \tag{3.3}$$

This is not a very simple product formula as it contains two infinite series, one of which (the A series) possessing negative contributions. Negative terms in any result always means that overcounting is occurring, which is inefficient. A far simpler result for Sp_{2k}, derived by Newell (1951) and later Littlewood (1958), is

$$Sp_{2k} \quad \langle\lambda\rangle \times \langle\mu\rangle = \sum_{\zeta} \langle(\lambda/\zeta)\cdot(\mu/\zeta)\rangle \quad . \tag{3.4}$$

This result is far more efficient than (3.3), containing only positive terms. Like (3.1),(3.2) and (3.3) the formula is rank-independent given that any non-standard labels that occur must be modified by the appropriate rule in table (2.6) for a particular k.

As an example of the use of (3.4) consider the product

$$\begin{aligned} Sp_{2k} \quad \langle 21\rangle \times \langle 1^2\rangle &= \langle 21.1^2\rangle + \langle(21/1).(1^2/1)\rangle \\ &\quad +\langle(21/1^2).(1^2/1^2)\rangle \\ &= \langle 21.1^2\rangle + \langle(2+1^2).1\rangle + \langle 1.0\rangle \\ &= \langle 32\rangle + \langle 31^2\rangle + \langle 2^2 1\rangle + \langle 21^3\rangle + \langle 3\rangle \\ &\quad + 2\langle 21\rangle + \langle 1^3\rangle + \langle 1\rangle \quad . \end{aligned} \tag{3.5}$$

This product is valid as it stands for all Sp_{2k}, $k \geqslant 4$. However for $k < 4$ some labels require modification.

For instance in the case k = 2 (Sp_4) the product becomes

$$Sp_4 \quad \langle 21\rangle \times \langle 1^2\rangle = \langle 32\rangle + \langle 3\rangle + \langle 21\rangle + \langle 1\rangle ,$$

as in Sp_4 $\langle 31^2\rangle$, $\langle 2^21\rangle$, $\langle 21^3\rangle$ and $\langle 1^3\rangle$ are all found to vanish, while $\langle 21^3\rangle$ is the negative of $\langle 21\rangle$.

Since the orthogonal groups contain two different types of irrep (tensor and spinor), consideration of O_n products is complicated by the fact that there are three different types of product: tensor-tensor, tensor-spinor, and spinor-spinor.

Derivation of the tensor-tensor product rule for O_n runs along similar lines to the Sp_{2k} rules, (3.3) and (3.4). Using (2.9) this rule is just (King 1975a)

$$O_n \qquad [\lambda] \times [\mu] = [((\lambda/C).(\mu/C))/D] \tag{3.6}$$

which, like (3.3) is inefficient. Once again the rule is easily bettered by the rule of Newell (1951) and Littlewood (1958), that

$$O_n \qquad [\lambda] \times [\mu] = \sum_{\zeta} [(\lambda/\zeta).(\mu/\zeta)] \quad . \tag{3.7}$$

The rules (3.4) and (3.7) are almost identical, the only difference being in the modification rules for a particular rank of group. The relationship between (3.3) and (3.4) and also (3.6) and (3.7) results in a set of identities for series which are considered in appendix I.

A general method for evaluating the products of O_n involving the spinor irreps has been developed by Butler and Wybourne (1969) and much simplified by King (1975a). Crucial to these results was the evaluation of the Kronecker square of the basic spinor irrep, $\Delta = [\Delta;0]$. This was first obtained by Brauer and Weyl (1935) and may be expressed as

$$O_{2k} \qquad \Delta^2 = [1^k] + \sum_{m=0}^{k-1} [1^m] + [1^m]^* = [Q] = [Q^*] \tag{3.8a}$$

$$O_{2k+1} \qquad \Delta^2 = \sum_{m=0}^{k} [1^m]^{(*)^m} = \tfrac{1}{2}[Q^*] \tag{3.8b}$$

The series Q^* and its inverse P^* are convenient generalisations of the series Q and P and are defined by

$$Q^* = \sum_m \{1^m\}^{(*)^m} \quad , \quad P^* = \sum_m (-1)^m \{m\}^{(*)^m} \ , \tag{3.9}$$

where it is understood that

$$[\lambda^{(*)^m}] = [\lambda]^{(*)^m} = \begin{cases} [\lambda] & m \text{ even} \\ [\lambda]^* & m \text{ odd} \ . \end{cases} \tag{3.10}$$

In equations (3.8), the final statement of the expressions contain apparently unbounded infinite series of S-functions. In reality bounds are supplied by the O_{2k} and O_{2k+1} modification rules (table 2.6) which place a limit on the number of parts in the irreps of the series.

To evaluate products involving a general spinor irrep of the form $[\Delta;\lambda]$ it is necessary to know how such an irrep can be expressed as a product of the basic spinor and a sum of tensor irreps. These relations and their inverses are given by (King 1975a, King et al. 1981)

$$O_{2k} \qquad \Delta \times [\lambda] = [\Delta;\lambda/Q] \tag{3.11a}$$

$$[\Delta;\lambda] = \Delta \times [\lambda/P] \tag{3.11b}$$

$$O_{2k+1} \qquad \Delta \times [\lambda] = [\Delta;\lambda/Q^*] \tag{3.12a}$$

$$[\Delta;\lambda] = \Delta \times [\lambda/P^*] \tag{3.12b}$$

Simple manipulations of (3.8), (3.11) and (3.12) along with the S-function series identity

$$(\lambda.\mu)/Z = (\lambda/Z).(\mu/Z) \tag{3.13}$$

true for series Z = P,Q,L,M,V and W, and discussed in appendix I, lead to the desired Kronecker product formulae. These are given by King (1975a) and King et al. (1981) as

$$O_{2k} \qquad [\Delta;\lambda] \times [\mu] = \sum_{\zeta} [\Delta;(\lambda/\zeta).(\mu/\zeta Q)] \tag{3.14a}$$

$$[\Delta;\lambda] \times [\Delta;\mu] = \sum_{\zeta} [(\lambda/\zeta).(\mu/\zeta).Q] \tag{3.14b}$$

$$O_{2k+1} \qquad [\Delta;\lambda] \times [\mu] = \sum_{\zeta} [\Delta;(\lambda/\zeta).(\mu/\zeta Q^*)] \tag{3.15a}$$

$$[\Delta;\lambda] \times [\Delta;\mu] = \sum_{\zeta} \tfrac{1}{2}[(\lambda/\zeta).(\mu/\zeta).Q^*] \quad . \tag{3.15b}$$

Like equation (3.8), formulae (3.14b) and (3.15b) contain series of S-functions bounded only by modification rules. In practice this creates a not insignificant overcounting problem in these formulae as the O_n modification rules do not provide a very efficient cut-off. Careful examination of the modification rules (table 2.6) shows that the highest possible value of m which will give non-vanishing terms in a product of the type $\sum_m [1^m.\lambda]$ is given by

$$O_n \qquad m_{max} = n + \lambda_1 \quad . \tag{3.16}$$

Various manipulations may be performed on (3.14b) and (3.15b) (King et al. 1981) so that fewer intermediate terms are produced. This is largely achieved by noticing that the modification rule can be applied earlier in the calculation. However the cost of this approach is in the simplicity of the equations. Nevertheless equations (3.14) and (3.15), along with (3.7), represent the easiest and most efficient methods for evaluating the products of the orthogonal groups.

For the subgroup SO_n the situation regarding products is more difficult. The problem lies in the reducibility of the self-associated irreps of O_n under restriction to SO_n. There is no difficulty for SO_{2k+1} since all irreps of O_{2k+1} remain irreducible on restriction to SO_{2k+1}. The equations (3.7) and (3.15) remain valid for SO_{2k+1} except that now the * is irrelevant and may be deleted from (3.15) and from the modification rule in table (2.6).

In the case of SO_{2k} the formulae (3.7) and (3.14) are valid but not always useful as they stand since the representations are all reducible except for the tensor representation, $[\lambda]$ with $\lambda_k = 0$. The decomposition of the reducible representations is given by (2.13) but it is convenient to use (2.15e) to rewrite (2.13a) as

$$O_{2k} \downarrow SO_{2k} \qquad [\square;\lambda] \downarrow [\square;\lambda]_+ + [\square;\lambda]_- \ . \tag{3.17}$$

Most progress towards a solution for products involving $[\square;\lambda]_\pm$ or $[\Delta;\lambda]_\pm$ has been made through the use of difference characters (Murnaghan 1938, Littlewood 1940). These are defined for tensor and spinor representations as

$$SO_{2k} \qquad [\lambda]'' = [\lambda]_+ - [\lambda]_- \tag{3.18a}$$

$$[\square;\lambda]'' = [\square;\lambda]_+ - [\square;\lambda]_- \tag{3.18b}$$

$$[\Delta;\lambda]'' = [\Delta;\lambda]_+ - [\Delta;\lambda]_- \ . \tag{3.18c}$$

Equations (2.13) and (3.17) can then be used to give

$$SO_{2k} \qquad [\lambda]_\pm = \tfrac{1}{2}([\lambda] \pm [\lambda]'') \tag{3.19a}$$

$$[\square;\lambda]_\pm = \tfrac{1}{2}([\square;\lambda] \pm [\square;\lambda]'') \tag{3.19b}$$

$$[\Delta;\lambda]_\pm = \tfrac{1}{2}([\Delta;\lambda] \pm [\Delta;\lambda]'') \ . \tag{3.19c}$$

Using (3.19), the Kronecker products of the SO_{2k} irreps may be obtained if the rules are known for the difference character. Butler and Wybourne (1969) and later King et al. (1981) have used this method to obtain many results for SO_{2k}. Vital to the success of this method was the finding of relations analog to (3.12). These are given by

$$SO_{2k} \quad [\square;\lambda] = \square \times [\lambda/Y] + 2 \sum_{s,t} (-1)^t [1^{k-1-t}] \times [\lambda/(1+t+s)/s] \tag{3.20a}$$

$$\square \times [\lambda] = [\square;\lambda/X] + 2 \sum_{s,t} [1^{k-1-t};\lambda/1^{1+t+s}/1^s] \tag{3.20b}$$

$$[\square;\lambda]'' = \square'' \times [\lambda/W] \tag{3.21a}$$

$$\square'' \times [\lambda] = [\square;\lambda/V]'' \tag{3.21b}$$

$$[\Delta;\lambda]'' = \Delta'' \times [\lambda/M] \tag{3.22a}$$

$$\Delta'' \times [\lambda] = [\Delta;\lambda/L]'' \quad . \tag{3.22b}$$

Also necessary are the ten Kronecker products for the four basic irreps, $\square$, $\square''$, Δ and Δ''. These are listed in full by King et al. (1981). Extensive manipulations produce the required product formulae between difference characters and also between difference characters and the ordinary characters. These formulae along with (3.14) and (3.7) make up a set of fifteen which may be used together according to (3.19) to give the Kronecker products of the irreps of SO_{2k}.

In practice, this is an extremely tedious procedure, and produces much overcounting. The method is also unsatisfying from a purely aesthetic point of view as it is complicated and there are no compact general formulae produced for the products of the irreps. Some general formulae have been produced (King et al. 1981) by algebraically combining the appropriate formulae together. However attempts to do this for the complete set of SO_{2k}

products are frustrated by the complicated nature of (3.20) and of some of the basic Kronecker products.

This is as far as this approach of using S-function manipulation currently can be taken. In spite of the obvious power of the method, there are still problems which have not been fully resolved. Firstly there are the products in the exceptional groups which have yet to be discussed. Secondly there is the question of whether more compact formulae exist for the products of the irreps of SO_{2k}. These problems may be resolved by the use of a new technique, discussed in the next section.

(3.2) THE BRANCHING RULE METHOD

When the group elements of a group G are restricted to those of a subgroup H of G, each irrep $\underset{\sim}{\lambda}_G$ of G yields a representation of H, which is, in general, reducible. The problem of resolving this representation into the irreps of H is called the branching rule problem. The solution to this problem is a branching rule, which may be written as

$$G{\downarrow}H \qquad \underset{\sim}{\lambda}_G \downarrow \sum_{\underset{\sim}{\sigma}_H} B_{\underset{\sim}{\lambda}_G}^{\underset{\sim}{\sigma}_H} \underset{\sim}{\sigma}_H \quad , \tag{3.23}$$

where $B_{\underset{\sim}{\lambda}_G}^{\underset{\sim}{\sigma}_H}$ is a non-negative integer called the branching multiplicity. Similarly an inverse rule can be defined

$$H{\uparrow}G \qquad \underset{\sim}{\sigma}_H \uparrow \sum_{\underset{\sim}{\lambda}_G} A_{\underset{\sim}{\sigma}_H}^{\underset{\sim}{\lambda}_G} \underset{\sim}{\lambda}_G \quad , \tag{3.24}$$

where $A^{\underset{\sim}{\lambda}_G}_{\underset{\sim}{\sigma}_H}$ may be a positive or negative integer. This process should not be confused with induction, which also uses the '$\uparrow$' symbol.

Solutions to the branching rule problem may be used to advantage in evaluating Kronecker products. In working with a group it is often useful to consider subgroups, or to consider the group as a subgroup of some larger group. The relationship between the products is such that if

$$\text{G} \qquad \underset{\sim}{\lambda}_G \times \underset{\sim}{\mu}_G = \sum_{\underset{\sim}{\nu}_G} K^{\underset{\sim}{\nu}_G}_{\underset{\sim}{\lambda}_G \underset{\sim}{\mu}_G} \underset{\sim}{\nu}_G \tag{3.24a}$$

and

$$\text{M} \qquad \underset{\sim}{\sigma}_H \times \underset{\sim}{\tau}_H = \sum_{\underset{\sim}{\rho}_H} K^{\underset{\sim}{\rho}_H}_{\underset{\sim}{\sigma}_H \underset{\sim}{\tau}_H} \underset{\sim}{\rho}_H \tag{3.24b}$$

then

$$K^{\underset{\sim}{\nu}_G}_{\underset{\sim}{\lambda}_G \underset{\sim}{\mu}_G} = \sum_{\underset{\sim}{\sigma}_H, \underset{\sim}{\tau}_H, \underset{\sim}{\rho}_H} B^{\underset{\sim}{\sigma}_H}_{\underset{\sim}{\lambda}_G} B^{\underset{\sim}{\tau}_H}_{\underset{\sim}{\mu}_G} K^{\underset{\sim}{\rho}_H}_{\underset{\sim}{\sigma}_H \underset{\sim}{\tau}_H} A^{\underset{\sim}{\nu}_G}_{\underset{\sim}{\rho}_H} \tag{3.25a}$$

$$K^{\underset{\sim}{\rho}_H}_{\underset{\sim}{\sigma}_H \underset{\sim}{\tau}_H} = \sum_{\underset{\sim}{\lambda}_G, \underset{\sim}{\mu}_G, \underset{\sim}{\nu}_G} A^{\underset{\sim}{\lambda}_G}_{\underset{\sim}{\sigma}_H} A^{\underset{\sim}{\mu}_G}_{\underset{\sim}{\tau}_H} K^{\underset{\sim}{\nu}_G}_{\underset{\sim}{\lambda}_G \mu_G} B^{\underset{\sim}{\rho}_H}_{\underset{\sim}{\nu}_G} \ . \tag{3.25b}$$

This relationship was in fact what was being exploited in section (3.1) to obtain products for O_n and Sp_{2k}. Equations (2.9) are just equations (3.23) and (3.24) for the special case $U_n \downarrow O_n$, while (2.10) gives the rule for $U_{2k} \downarrow Sp_{2k}$. Thus the Kronecker products in Sp_{2k} and O_n may be obtained from the Kronecker products in U_{2k} and U_n by means of (3.25b). Similarly Wybourne and Bowick (1977) used both (3.25a) and (3.25b) to obtain Kronecker products for the exceptional groups. They used the $SO_7 \downarrow G_2$ branching rule with (3.25b) to obtain the G_2 products, and the $F_4 \downarrow SO_9$, $E_6 \downarrow SU_2 \times SU_6$, $E_7 \downarrow SU_8$, and $E_8 \downarrow SU_9$ branching rules with (3.25a) to obtain the F_4, E_6, E_7 and E_8 products respectively. The exceptional group products are harder to evaluate than the classical group products because the branching rules are very much harder to produce and cannot be stated as a general formulae such as in (2.9) and (2.10).

Another method of evaluating Kronecker products of Lie groups is the use of weight space techniques. An early technique was to decompose the two irreps into two sums of weights, multiply the weights together in a trivial operation and then convert the sum of weights into a sum of irreps. This procedure is a special case of the use of (3.25a), where the subgroup is the product group $U_1^{\times k}$ consisting of k U_1 groups. Racah (1964) and Speiser (1964) have much simplified this method so that only one irrep need be decomposed into weights. Special modification rules are needed to standardise labels in this method, and these are just those rules given in table (2.7).

King (1981a) has generalised Racah and Speiser's result so that any subgroup of rank k, naturally embedded in G, may be used, instead of just $U_1^{\times k}$. The method may be expressed as a formula

$$G \qquad \underset{\sim}{\lambda}_G \times \underset{\sim}{\mu}_G = \sum_{\underset{\sim}{\sigma}_H \underset{\sim}{\rho}_H} B^{\underset{\sim}{\sigma}_H}_{\underset{\sim}{\lambda}_G} \; K^{\underset{\sim}{\rho}_H}_{\underset{\sim}{\sigma}_H}, (\underset{\sim}{\mu}_G + \underset{\sim}{\delta}_G - \underset{\sim}{\delta}_H)_H (\underset{\sim}{\rho}_H - \underset{\sim}{\delta}_G + \underset{\sim}{\delta}_H)_G \; , \tag{3.26}$$

where $\underset{\sim}{\delta}_G$ and $\underset{\sim}{\delta}_H$ are half the sums of the positive roots of

G and H respectively (King and Al-Qubanchi 1981a). The notation $(\underset{\sim}{\rho}_H - \delta_G + \delta_H)$ is just short for

$$(\underset{\sim}{\rho}_H - \underset{\sim}{\delta}_G + \underset{\sim}{\delta}_H) = (\underset{\sim}{\rho}_{1H}-\underset{\sim}{\delta}_{1G}+\underset{\sim}{\delta}_{1H}, \underset{\sim}{\rho}_{2H}-\underset{\sim}{\delta}_{2g}+\underset{\sim}{\delta}_{2H}, \cdots , \underset{\sim}{\rho}_{kH}-\underset{\sim}{\delta}_{kG}+\underset{\sim}{\delta}_{kH}). \qquad (3.27)$$

The G and H subscripts in (3.26) specify to which groups the labels are to be standardised.

To use the method four things must be known about the group G and the chosen subgroup H: firstly the branching rule for G↓H must be known for the irreps $\underset{\sim}{\lambda}_G$; secondly the term $\underset{\sim}{\delta}_G-\underset{\sim}{\delta}_H$; thirdly simple and efficient means of evaluating the Kronecker products of the irreps of H; and finally the modification rules in the group G, to make the terms $(\underset{\sim}{\rho}_H-\underset{\sim}{\delta}_G+\delta_H)$ G-standard.

The method may be used to evaluate Kronecker products in both classical and exceptional groups, and is of particular interest for the cases where the methods of section (3.1) have not been successful.

(3.3) THE BRANCHING RULE $SO_{2k} \downarrow U_k$

Before considering the question of the SO_{2k} products the subgroup to be used must be chosen and the branching rule from SO_{2k} must be evaluated. As is seen in the next section, considerable simplification to (3.26) may be achieved by using a unitary group as the subgroup, and so the branching rule to be determined in the case of SO_{2k} is $SO_{2k} \downarrow U_k$.

This branching has proved difficult to determine because of the reducibility of the self-associate irreps of O_{2k} on reduction to SO_{2k}, the same reason that made the Kronecker products difficult to evaluate in section (3.1). However in this case a general formula can be derived.

Important in the derivations are the series identities

$$Z \times \{\bar{\mu};\lambda\} = \{\overline{\mu/Z};\lambda\cdot Z\} \tag{3.27a}$$

$$\bar{Z} \times \{\bar{\mu};\lambda\} = \{\overline{\mu\cdot Z};\lambda/Z\} \tag{3.27b}$$

for the series Z = L,M,P,Q,V and W, and

$$\sum_{\zeta} \{\zeta/Z;(\lambda/\zeta)\cdot\zeta\} = \sum_{\eta} \{\bar{\eta};(\lambda/\eta\cdot Z)\cdot\rho\} \tag{3.28}$$

for any series Z. Proofs for these identities are found in appendix I.

King (1975b) has given the branching rules for $O_{2k}\downarrow U_k$, and while they are not generally applicable to the SO_{2k} branching, they may be used in conjunction with difference character techniques to derive the $SO_{2k}\downarrow U_k$ rules. The tensor branching is given by

$$O_{2k}\downarrow U_k \qquad [\lambda] \downarrow \sum_{\zeta} \{\bar{\zeta};\lambda/\zeta\cdot B\} , \tag{3.29}$$

which is valid for $SO_{2k}\downarrow U_k$ if the number of parts of (λ) is less than k. The rule for the spinor irrep, $[\Delta;\lambda]$, is also given by King (1975b); this may be derived once the branching rule for the basic spinor, Δ, is known.

Since an almost identical derivation gives the branching rule for $[\Delta;\lambda]''$ and $[\square;\lambda]''$, the derivation is given here in full. The decomposition of the basic spinor is given by

$$O_{2k}\downarrow U_k \qquad \Delta\downarrow\varepsilon^{\frac{1}{2}} \times Q = \varepsilon^{-\frac{1}{2}} \times \bar{Q} \tag{3.30}$$

The half-integral powers of ε correspond to double-valued representations of U_n, as mentioned in chapter 2. The alternate form of the branching arises because the series Q is made up of only one column partitions and hence $\bar{Q} = \varepsilon^{-1} \times Q$. The general rule can then be evaluated using (3.11b) followed by (3.29) and (3.30) along with the identities (3.27a) and (3.28)

$$O_{2k}\downarrow U_k \qquad [\Delta;\lambda] = \Delta \times [\lambda/P]$$

$$\downarrow \varepsilon^{-\frac{1}{2}} \times Q \times \sum_{\zeta} \{\bar{\zeta};\lambda/\zeta\cdot B\cdot P\}$$

$$= \varepsilon^{-\frac{1}{2}} \times \sum_{\zeta} \{\overline{\zeta/Q};(\lambda/\zeta\cdot B\cdot P)\cdot Q\}$$

$$= \varepsilon^{-\frac{1}{2}} \times \sum_{\eta} \{\bar{\eta};(\lambda/\zeta\cdot B\cdot P\cdot Q)\cdot Q\}$$

$$= \varepsilon^{-\frac{1}{2}} \times \sum_{\eta} \{\bar{\eta};(\lambda/\zeta\cdot B)\cdot Q\} \tag{3.31a}$$

Use of the alternate form of the branching of the basic spinor produces the alternate general branching rule

$$O_{2k}\downarrow U_k \qquad [\Delta;\lambda] \downarrow \varepsilon^{\frac{1}{2}} \times \sum_{\eta} \{\overline{\eta\cdot Q};\lambda/\zeta\cdot B\} \tag{3.31b}$$

The rules (3.29) and (3.31) may be used to derive the $SO_{2k} \downarrow U_k$ rules for $[\Delta;\lambda]_\pm$ and $[\lambda]_\pm$ only if the decompositions for $[\Delta;\lambda]''$ and $[\lambda]''$ are known. Considering the spinor case first, the decomposition of the Δ'', basic spinor irrep is given by

$$SO_{2k}\downarrow U_k \qquad \Delta'\downarrow(-1)^k\varepsilon^{-\frac{1}{2}}\times L = \varepsilon^{\frac{1}{2}} \times L \quad . \tag{3.32}$$

Using (3.22a), (3.29) and (3.32) a result analagous to (3.31) may be produced

$$SO_{2k}\downarrow U_k \qquad [\Delta;\lambda]''\downarrow\varepsilon^{-\frac{1}{2}} \times \sum_{\zeta} \{\bar{\zeta};(\lambda/\zeta\cdot B)\cdot(-1)^k L\} \tag{3.33a}$$

$$[\Delta;\lambda]''\downarrow\varepsilon^{\frac{1}{2}} \times \sum_{\zeta} \{\overline{\zeta\cdot L};\lambda/\zeta B\} \quad . \tag{3.33b}$$

Using (3.19c) the two rules (3.31) and (3.33) may be combined to produce the rules for $[\Delta;\lambda]_\pm$

$$SO_{2k}\downarrow U_k \qquad [\Delta;\lambda]_\pm\downarrow\varepsilon^{-\frac{1}{2}} \times \sum_{\zeta} \{\bar{\zeta};(\lambda/\zeta\cdot B)\cdot Q_{\pm(-1)^k}\} \tag{3.34a}$$

$$[\Delta;\lambda]_\pm\downarrow\varepsilon^{\frac{1}{2}} \times \sum_{\zeta} \{\overline{\zeta\cdot Q_\pm};\lambda/\zeta\cdot B\} \quad , \tag{3.34b}$$

where the series $Q_\pm$, defined by

$$Q_\pm = \tfrac{1}{2}(Q\pm L), \tag{3.35}$$

are given by

$$Q_+ = \sum_m \{1^{2m}\} \tag{3.36a}$$

$$Q_- = \sum_m \{1^{2m+1}\} \quad . \tag{3.36b}$$

Turning to the tensor difference character $[\lambda]''$, a formula may be derived in exactly the same way as for the spinor difference character if the notation $[\Box;\lambda]''$ is used. The appropriate decomposition of the basic $\Box''$ irrep is

$$SO_{2k}\downarrow U_k \qquad \Box''\downarrow(-1)^k\varepsilon^{-1} \times V = \varepsilon \times \bar{V} \quad . \tag{3.37}$$

This gives, by virtue of (3.21a) and (3.29),

$$SO_{2k}\downarrow U_k \qquad [\Box;\lambda]''\downarrow\varepsilon^{-1} \times \sum_{\zeta} \{\bar{\zeta};(\lambda/\zeta B)\cdot(-1)^k V\} \tag{3.38a}$$

$$[\Box;\lambda]''\downarrow\varepsilon \quad \times \sum_{\zeta} \{\overline{\zeta\cdot V};\lambda/\zeta B\} \tag{3.38b}$$

However this formula for the difference character is not of a form that can be easily combined with (3.29), and a formula for $[\Box;\lambda]$ is required before (3.28) may be used to produce a formula for $[\lambda]_{\pm}$ or $[\Box;\lambda]_{\pm}$.

The decomposition for the $\Box$ irrep may be easily derived from (3.29), and is

$$SO_{2k}\downarrow U_k \qquad \Box\downarrow\varepsilon^{-1} \times X = \varepsilon \times \bar{X} \quad . \tag{3.39}$$

Unfortunately the general result cannot be obtained from this decomposition in the same manner as before. This is due to the fact that (3.20a) is so complicated and also because, unlike P and Q, and V and W, X and Y are not inverse series. However in spite of all this a general formula for $[\Box;\lambda]$ may be found, and is in fact remarkably simple, considering the complexity of (3.20a). The formula is

$$SO_{2k}\downarrow U_k \qquad [\square;\lambda]\downarrow\varepsilon^{-1} \times \sum_{\zeta} \{\overline{\zeta};(\lambda/\zeta B)\cdot X\} \tag{3.40a}$$

$$[\square;\lambda]\downarrow\varepsilon \quad \times \sum_{\zeta} \{\overline{\zeta\cdot X};\lambda/\zeta B\} \ . \tag{3.40b}$$

This formula was originally found by a series of 'inspired guesses' and much use of the program SCHUR (chapter 4). The first complete and rigorous proof is due to King (private communication). To establish this proof, some identities must first be considered. Firstly a result concerning the $(1^m;\lambda)$ notation of section (2.1) from King et al. (1981)

$$\{1^s\}\cdot\{p\} = \sum_{t} \{1^{s+t};p/1^t\} \tag{3.41}$$

Secondly from (2.7) it may be seen that

$$PQ = \sum_{p,s} (-1)^p\{p\}\cdot\{1^s\} = \{0\} = 1 \quad . \tag{3.42}$$

This summation may be broken up into a summation over different m = p+s, in which only the m = 0 term contributes, i.e.

$$\sum_{m} \sum_{\substack{p,s \\ p+s=m}} (-1)^p\{p\}\cdot\{1^s\} = \begin{cases} \{0\} = 1 & m=0 \\ 0 & m\neq 0 \end{cases} \ . \tag{3.43}$$

The first step of the proof is to break up the $[\square;\lambda]$ into a sum of products of an irrep with a one-column label with a general label. This is given by

$$SO_{2k} \qquad [\square;\lambda] = \sum_{p,q} (-1)^{p+q}[1^{k+p-q}]\times[\lambda/p.q] \tag{3.44}$$

The proof of this is

$$\sum_{p,q} (-1)^{p+q}[1^{k+p-q}]\times[\lambda/p.q]$$

$$= \sum_{p,q,t} (-1)^{p+q}[(1^{k+p-q-t}).(\lambda/p.q.1^t)] \quad \text{by (3.7)},$$

$$= \sum_{p,q,t,s} (-1)^{p+q}[1^{k+p-q-t+s};\lambda/p.q.1^t.1^s] \quad \text{by (3.41)},$$

$$= [1^k;\lambda] \qquad \text{by (3.43) used twice} .$$

$$= [\square;\lambda] \quad .$$

The next step is to find the branching rule for each term in (3.44). Both terms may be branched using (3.29), however the first term is a special case and its branching may be simplified. For $[1^{k+p-q}]$ the formula (3.29) gives

$$SO_{2k}{\downarrow}U_k \qquad [1^{k+p-q}] \downarrow \sum_{s,t} \{\overline{1^t};1^{k+p-q-t-2s}\} \quad .$$

By making the substitutions $t = k+p-u-v$ and $t+2s = k+p - u+v$ this is just

$$\sum_{u,v} \{\overline{1^{k+p-u-v}};\ 1^{u-q-v}\}$$

$$= \sum_{u} \{\overline{1^{k+p-u}}\} \times \{1^{u-q}\} \qquad \text{by (3.2)} \quad ,$$

$$= \varepsilon^{-1} \times \sum_{u} \{1^{u-p}\} \times \{1^{u-q}\} \quad \text{by (2.12)} \quad ,$$

giving in summary that

$$SO_{2k}\downarrow U_k \qquad [1^{k+p-q}]\downarrow\varepsilon^{-1} \times \sum_u \{1^{u-p}\} \times \{1^{u-q}\} \ . \tag{3.45a}$$

A similar sequence of operations gives

$$SO_{2k}\downarrow U_k \qquad [1^{k+p-q}]\downarrow\varepsilon \times \sum_u \{\overline{1^{p-u}}\} \times \{\overline{1^{q-u}}\} \quad . \tag{3.45b}$$

The branching rule for $[\square;\lambda]$ can now be given by

$$SO_{2k}\downarrow U_k \qquad [\square;\lambda]\downarrow\varepsilon^{-1} \times \sum_{p,q,u} (-1)^{p+q}\{1^{u-p}\} \times \{1^{u-q}\}$$

$$\times \sum_{\xi} \{\overline{\xi};\lambda/p.q.\xi.B\}$$

$$= \varepsilon^{-1} \times \sum_{\substack{p,q,u,\\ \xi,v,w}} (-1)^{p+q}\{\overline{\xi.1^v 1^w};$$

$$(\lambda/p.q.\xi.B).1^{u-p-v}.1^{u-q-w}\}$$

by (3.2)

$$= \varepsilon^{-1} \times \sum_{\substack{p,q,u,\\ \zeta,v,w}} (-1)^{p+q}\{\overline{\zeta};$$

$$(\lambda/p.q.\zeta.B.1^v.1^w).1^{u-p-v}.1^{u-q-w}\}$$

by (3.28)

$$= \varepsilon^{-1} \sum_{u,\zeta} \{\overline{\zeta};(\lambda/\zeta B).1^u.1^u\}$$

by (3.43)

Noting the fact that the X series is just $\sum_n \{1^n\}\cdot\{1^n\}$, the general result may now be written as (3.40a). The proof for (3.40b) follows a similar derivation.

The branching for the SO_{2k} irrep $[\square;\lambda]_{\pm}$ may now be obtained using (3.38) and (3.40) combined with (3.19b)

$$SO_{2k}\downarrow U_k \qquad [\square;\lambda]_{\pm}\downarrow\varepsilon^{-1} \times \sum_{\zeta} \{\bar{\zeta};(\lambda/\zeta B).X_{\pm(-1)^k}\} \tag{3.46a}$$

$$[\square;\lambda]_{\pm}\downarrow\varepsilon \times \sum_{\zeta} \{\overline{\zeta.X_{\pm}};\lambda/\zeta B\} \tag{3.46b}$$

where the series $X_{\pm}$, defined by

$$X_{\pm} = \tfrac{1}{2}(X \pm V),$$

are given by

$$X_{+} = \sum_{m,p} \{2^{2m-2p}1^{2p}\} \tag{3.48a}$$

$$X_{-} = \sum_{m,p} \{2^{2m+1-2p}1^{2p}\} \quad . \tag{3.48b}$$

Thus (3.29), (3.34) and (3.46) are the complete set of branching rules for $SO_{2k}\downarrow U_k$. A table of these rules for the physically interesting case $SO_{10}\downarrow U_5$ (or, more accurately $SO_{10}\downarrow U_1 \times SU_5$) is given for all irreps of up to power four.

(3.4) KRONECKER PRODUCTS FOR SO_{2k}

Before the products can be evaluated it is necessary to know the terms, $\delta_{SO_{2k}}$ and δ_{U_k}. These terms, half the sums of the positive roots of the two groups, were evaluated for this labelling scheme by King and Al-Qubanchi (1981a).

Table (3.1) gives these terms for both the classical and exceptional Lie groups. From this table, the difference of the two terms $\delta_{SO_{2k}}$ and δ_{U_k} is seen to be trivial

$$\delta_{SO_{2k}} - \delta_{U_k} = (0,0, \ldots , 0) \ . \tag{3.49}$$

This fact can be used to produce significant simplification to the branching rule method formula (3.26) for this special case with $G = SO_{2k}$ and $H = U_k$. It now can be written as

$$(\underset{\sim}{\lambda} \times \underset{\sim}{\mu})_G = \sum_{\underset{\sim}{\sigma}_H, \underset{\sim}{\rho}_H} B^{\underset{\sim}{\sigma}_H}_{\underset{\sim}{\lambda}_G} \, K^{\underset{\sim}{\rho}}_{\underset{\sim}{\sigma}_H} (\underset{\sim}{\mu}_G)_H \, (\underset{\sim}{\rho}_H)_G \ . \tag{3.50}$$

The simplicity of the equations is due to a good choice of the subgroup, U_k, which is also good for fast evaluations of Kronecker products.

The only remaining question to be dealt with before the general products may be expressed is how to handle the $\{\square;\lambda\}_{\pm}$ and $\{\Delta;\lambda\}_{\pm}$ labels in U_k. This may be resolved using the special irrep, ε. Equation (2.15e) indicates that

$$U_k \qquad \{\square;\lambda\} = \varepsilon \times \{\lambda\} \ , \tag{3.51a}$$

while (2.15c) suggests

$$U_k \qquad \{\Delta;\lambda\} = \varepsilon^{\frac{1}{2}} \times \{\lambda\} \ . \tag{3.51b}$$

Equation (2.16), which gives

TABLE (3.1): Half the sum of positive roots

Group	$\underset{\sim}{\delta}_G$
SU_{k+1}	$(k,\ k-1,\ \ldots\ ,\ 2,1)$
SO_{2k+1}	$(k-\frac{1}{2},\ k-\frac{3}{2},\ \ldots\ ,\ \frac{3}{2},\frac{1}{2})$
Sp_{2k}	$(k,\ k-1,\ \ldots\ ,\ 2,1)$
SO_{2k}	$(k-1,\ k-2,\ \ldots\ ,\ 1,0)$
G_2	$(3,1)$
F_4	$(\frac{11}{2},\ \frac{5}{2},\ \frac{3}{2},\ \frac{1}{2})$
E_6	$(11,\ 5,\ 4,\ 3,\ 2,\ 1)$
E_7	$(17,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1)$
E_8	$(29,\ 7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1)$

$$U_k \qquad \{\lambda\}_- = \{\overline{\lambda_k};\lambda_1,\lambda_2, \cdots, \lambda_{k-1}\} \tag{3.52}$$

leads then to the rules for $\{\square;\lambda\}_-$ and $\{\Delta;\lambda\}$,

$$U_k \qquad \{\square;\lambda\}_- = \varepsilon \times \{\overline{\lambda_k+2};\lambda_1,\lambda_2, \cdots, \lambda_{k-1}\} \tag{3.53a}$$

$$\{\Delta;\lambda\}_- = \varepsilon^{\frac{1}{2}} \times \{\overline{\lambda_k+1};\lambda_1,\lambda_2, \cdots, \lambda_{k-1}\} \ . \tag{3.53b}$$

The rules for $\{\square;\lambda\}_+$ and $\{\Delta;\lambda\}_+$ are the same as for (3.51).

The product rules for SO_{2k} can now be formulated in a succinct manner. Using (3.29) in (3.50), and noting (3.51) the following rules may be derived:

$$SO_{2k} \qquad [\lambda] \times [\mu]_+ = \sum_\zeta [\{\bar\zeta;\lambda/\zeta B\} \times \{\mu\}]_+ \tag{3.54a}$$

$$[\lambda] \times [\square;\mu]_+ = \sum_\zeta [\varepsilon\times\{\bar\zeta;\lambda/\zeta B\}\times\{\mu\}]_+ \tag{3.54b}$$

$$[\lambda] \times [\Delta;\mu]_+ = \sum_\zeta [\varepsilon^{\frac{1}{2}}\times\{\bar\zeta;\lambda/\zeta B\}\times\{\mu\}]_+ \tag{3.54c}$$

Similarly, using the branching rules (3.34) and (3.46)

$$SO_{2k} \qquad [\Delta;\lambda]_\pm \times [\mu]_+ = \sum_\zeta [\varepsilon^{\frac{1}{2}} \times\{\overline{\zeta.Q_\pm};\lambda/\zeta B\}\times\{\mu\}]_+ \tag{3.54d}$$

$$[\Delta;\lambda]_\pm \times [\square;\mu]_+ = \sum_\zeta [\varepsilon^{\frac{1}{2}} \times\{\bar\zeta;(\lambda/\zeta B).Q_{\pm(-1)^k}\}\times\{\mu\}]_+ \tag{3.54e}$$

$$[\Delta;\lambda]_\pm \times [\Delta;\mu]_+ = [\{\bar\zeta;(\lambda/\zeta B).Q_{\pm(-1)^k}\}\times\{\mu\}]_+ \tag{3.54f}$$

$$[\square;\lambda]_\pm \times [\mu]_+ = \sum_\zeta [\varepsilon\times\{\overline{\zeta.X_\pm};\lambda/\zeta B\}\times\{\mu\}]_+ \tag{3.54g}$$

$$[\square;\lambda]_{\pm} \times [\square;\lambda]_{+} = \sum_{\zeta}[\{\bar{\zeta};(\lambda/\zeta B).X_{\pm(-1)^k}\}\times\{\mu\}]_{+} \qquad (3.54h)$$

$$[\square;\lambda]_{\pm} \times [\Delta;\lambda]_{+} = \sum_{\zeta}[\varepsilon^{\frac{3}{2}}\times\{\overline{\zeta.X_{\pm}};\lambda/\zeta B\}\times\{\mu\}]_{+} \ . \qquad (3.54i)$$

In using these formulae, after all U_k modification is completed, the double partition labels for SO_{2k} must be modified to standard labels (table 2.1) by means of the weight-space-derived modification rules discussed in section (2.3) and presented in table (2.8).

The corresponding rules for the case where the second term on the left has a '-' subscript instead of a '+' may all be obtained by using (3.52) and (3.53) instead of (3.51). However the simplest way to obtain these products is to apply the involutory outer automorphism ,†, (2.14) to both sides of the rules.

In all cases where the number of parts, of $[\lambda]_{\pm}$ is less than k, $[\lambda]_{\pm} = [\lambda]$.

Since Kronecker products are commutative there are some redundancies in (3.54), moreover there are redundancies because of the one-to-one correspondence between the $[\square;\lambda]_{\pm}$ and the $[\nu]_{\pm}$ labels where (ν) has k parts. This allows a certain amount of choice in the use of (3.54). The most efficient complete set of formulae for the products of SO_{2k} is (3.7),(3.54a),(3.54c),(3.54e),(3.54f) and (3.54h).

The relationship between (3.54a) and (3.54b), say, hints at an additional benefit of using U_k as the subgroup in which the products are performed. In the second terms on the left hand side of the formulae (3.54), the (μ) may be replaced by the label of any associated irrep of $\{\mu\}$

in U_k if the term inside the square brackets on the right hand side is multiplied by the power of ε which associates the two irreps of U_k. For example, the formulae (3.54a) and (3.54b) may be generalised to

$$SO_{2k} \quad [\lambda] \times [m^k;\mu]_+ = \sum_{\zeta} [\varepsilon^m \times\{\bar{\zeta};\lambda/\zeta B\}\times\{\mu\}]_+ \quad , \tag{3.55}$$

where the $(1^k;\mu)$ notation has been generalised in a natural way to apply to multiple columns. The power of ε, m, may be any positive integer or half-integer, where a half-integer would indicate a spinor index. All reference to U_k may be dropped in (3.54) by employing (3.2) to give the analogues of (3.54a,c,e,f and h)

$$SO_{2k} \quad [\lambda] \times [\mu]_+ = \sum_{\zeta,\eta} [\bar{\zeta};(\lambda/\zeta.\eta.B).(\mu/\eta)]_+ \tag{3.56a}$$

$$[\lambda] \times [\Delta;\mu]_+ = \sum_{\gamma,\eta} [\Delta;\bar{\zeta};(\lambda/\zeta.\eta.B).(\mu/\eta)]_+ \tag{3.56b}$$

$$[\Delta;\lambda]_\pm \times [\square;\mu]_+ = \sum_{\zeta,\eta} [\Delta;\bar{\zeta};(\lambda/\zeta.\eta.B).(\mu/\eta).Q_{\pm(-1)^k}]_+ \tag{3.56c}$$

$$[\Delta;\lambda]_\pm \times [\Delta;\mu]_+ = \sum_{\zeta,\eta} [\bar{\zeta};(\lambda/\zeta.\eta.B).(\mu/\eta).Q_{\pm(-)^k}]_+ \tag{3.56d}$$

$$[\square;\lambda]_\pm \times [\square;\lambda]_+ = \sum_{\zeta,\eta} [\bar{\zeta};(\lambda/\zeta.\eta.B).(\mu/\eta).X_{\pm(-1)^k}]_+ , \tag{3.56e}$$

where (3.28) has been used. The only difficulty with (3.55) is that is is hard to see where to start. The fact that there are now no U_k labels hasn't completely done away with the need to consider U_k as the U_k modification rule in table (2.6) must still be used before employing the special rules of table (2.8).

It may be instructive to consider some practical examples. Firstly the product $[21] \times [21^3]_+$ in SO_8. This product may be evaluated using (3.54a), (3.54b) or (3.55a). The most efficient would seem to be (3.54a). The first step is to evaluate $\sum_{\zeta} \{\bar{\zeta};21/\zeta B\}$ in U_4, which may be done simply using Butler's tabulation of S-function outer products and divisions (Wybourne 1970). This gives

$$\sum_{\zeta} \{\bar{\zeta};21/\zeta B\} = \{\bar{0};21/B\} + \{\bar{1};2/B\} + \{\bar{1};1^2/B\}$$

$$+ \{\overline{1^2};1/B\} + \{\bar{2};1/B\} + \{\overline{21};0/B\}$$

$$= \{\bar{0};21\} + \{\bar{0};1\} + \{\bar{1};2\} + \{\bar{1};0\} + \{\bar{1};1^2\}$$

$$+ \{\bar{1}^2;1\} + \{\bar{2};1\} + \{\overline{21};0\}$$

This sum must now be multiplied by $\{21^3\}$ in U_4, giving

$$\{421^2\} + \{41^2\} + \{3^21^2\} + \{32^21\} + 2\{321\} + 2\{31^3\}$$

$$+\{\bar{1};31^2\} + \{31\} + \{2^3\} + 2\{2^21^2\} + \{\bar{1};2^21\} + \{2^2\}$$

$$+ 3\{21^2\} + \{\bar{1};21\} + \{1^4\} + \{\bar{1};1^3\} + \{1^2\} .$$

These labels must be made SO_8-standard using the modifications of table (2.8) to give

$$[21] \times [21^3]_+ = [421^2]_+ + [41^2] + [3^21^2]_+ + [32^21]_+$$
$$+ 2[321] + 2[31^3]_+ + [31^3]_- + [31] + [2^3]$$
$$+ 2[2^21^2]_+ + [2^21^2]_- + [2^2] + 3[21^2]$$
$$+ [1^4]_+ + [1^4]_- + [1^2] \quad .$$

Noting (3.55), the products $[21] \times [32^3]_+$, $[21] \times [43^3]_+$, $[21] \times [54^3]_+$ ... may be obtained by multiplying the sum of U_k irreps by ε^m before this last modification. For instance, the product $[21] \times [43^3]_+$ may be obtained by multiplying the sum by ε^2 to produce

$$\{643^2\} + \{63^22\} + \{5^23^2\} + \{54^23\} + 2\{5432\}$$
$$+ 2\{53^3\} + \{53^21\} + \{532^2\} + \{4^32\} + 2\{4^23^2\}$$
$$+\{4^231\} + \{4^22^2\} + 3\{43^22\} + \{3^4\} + \{3^31\}$$
$$+\{3^22^2\} \quad .$$

All these U_4 labels are also SO_8-standard so that

$$[21] \times [43^3]_+ = [643^2]_+ + [63^22]_+ + [5^23^2]_+ + [54^23]_+$$
$$+ 2[5432]_+ + 2[53^3]_+ + [53^21]_+ + [532^2]_+ + [4^32]_+$$
$$+ 2[4^23^2]_+ + [4^231]_+ + [4^22^2]_+$$
$$+ 3[43^22]_+ + [3^4]_+ + [3^31]_+ + [3^22^2]_+ \quad .$$

The products $[21] \times [21^3]_-$ and $[21] \times [43^3]_-$ can be obtained by applying $\dagger$ to both sides of these products, i.e. by interchanging the '+' and '-' subscripts.

As a second example consider $[\Delta;1]_- \times [2^21]_+$ in SO_6. This may be evaluated by (2.54d),(2.54e) or (2.56c). Using (2.54e) with k = 3 the first thing to calculate is

$$\sum_{\zeta} \{\bar{\zeta};(1/\zeta.B).Q_+\} = \sum_{\zeta,m} \{\bar{\zeta};(1/\zeta).1^{2m}\}$$

$$= \sum_m \{\bar{0};1.1^{2m}\} + \sum_m \{\bar{1};1^{2m}\}$$

$$= \{\bar{0};1\} + \{\bar{0};21\} + \{\bar{0};1^3\} + \{\bar{1};0\}$$

$$+ \{\bar{1},1^2\}$$

This must now be multiplied by $\{1^2\} = \varepsilon^{-1}\{2^21\}$ in U_3 to give

$$\{32\} + \{31^2\} + \{2^21\} + \{\bar{1};2^2\} + 2\{21\} + \{1^3\}$$
$$+ \{\bar{1};1^2\} + \{1\}$$

This sum may now be modified using the SO_{2k} spinor modification of table (2.8) producing

$$[\Delta;1]_- \times [2^21]_+ = [\Delta;32]_+ + [\Delta;31^2]_+ + [\Delta;2^21]_+$$

$$+ [\Delta;2^2]_- + 2[\Delta;21]_+ + [\Delta;1^3]_+$$

$$+ [\Delta;1^2]_- + [\Delta;1]_+ \quad .$$

Once again it is easy to derive the result $[\Delta;1]_+ \times [2^21]_-$ by applying $\dagger$, (2.14), to both sides.

As a final example consider the use of (3.54h) to obtain the product $[21^4]_- \times [21^4]_+$ in SO_{10}. Noting the equivalence $[21^4]_- \equiv [\square;1]_-$ in SO_{10}, the first step is to evaluate the branching rule

$$\sum_{\zeta} \{\bar{\zeta};(1/\zeta B).X_+\} = \sum_{\zeta,m,p} \{\bar{\zeta};(1/\zeta).(2^{2m-2p}1^{2p})\}$$

$$= \sum_{m,p} \{\bar{1};2^{2m-2p}1^{2p}\} + \sum_{m,p} \{1.(2^{2m-2p}1^{2p})\}$$

The U_5 modification rules restrict the size of m to be less than four in the first term and less than three in the second. The sum is then just

$$\{32^3\} + \{321^2\} + \{32\} + \{2^31\} + \{2^21\} + \{21^3\} + \{21\}$$
$$+\{1^3\} + \{1\} + \{\bar{1};2^4\} + \{\bar{1};2^21^2\} + \{\bar{1};1^4\} + \{\bar{1};2^2\}$$
$$+\{\bar{1};1^2\} + \{\bar{1};0\} \quad ,$$

which must be multiplied by $\{1\}$ in U_5, to give

$$\{42^3\} + \{421^2\} + \{42\} + \{3^22^2\} + \{3^21^2\} + \{3^2\} + \{32^31\}$$
$$+2\{32^21\} + \{321^3\} + 2\{321\} + \{31^3\} + \{31\} + 2\{2^4\}$$
$$+ \{2^31^2\} + \{2^3\} + 3\{2^21^2\} + 2\{2^2\} + \{21^4\} + 2\{21^2\}$$
$$+ \{2\} + 2\{1^4\} + 2\{1^2\} + \{0\} + \{\bar{1};32^3\} + \{\bar{1};321^2\}$$
$$+ \{\bar{1};32\} + \{\bar{1};2^31\} + \{\bar{1};2^21\} + \{\bar{1};21^3\} + \{\bar{1};21\}$$
$$+ \{\bar{1};1^3\} + \{\bar{1};1\} \quad .$$

After modification to SO_{10}-standard labels the final result is produced as

$$[21^4]_- \times [21^4]_+ = [42^3] + [421^2] + [42] + [3^22^2]$$
$$+ [3^21^2] + [3^2] + [32^31]_+ + [32^31]_- + 2[32^21]$$
$$+ [321^3]_+ + [321^3]_- + 2[321] + [31^3] + [31]$$
$$+ 2[2^4] + [2^31^2]_+ + [2^31^2]_- + [2^3] + 3[2^21^2]$$
$$+ 2[2^2] + [21^4]_+ + [21^4]_- + 2[21^2] + [2] + 2[1^4]$$
$$+ 2[1^2] + [0] \quad .$$

The example illustrates the fact that, for higher rank groups, the number of terms in a product increases quite quickly with weight. To keep the calculation as simple as possible it is always best to branch the smallest weight partition and multiply by the larger, although it is usually more efficient to branch down an irrep labelled by $[\lambda]$ in preference to one labelled by $[\Delta;\lambda]_\pm$. Normally the only occasion in which it is necessary or desirable to branch an irrep of the form $[\Box;\lambda]_\pm$ is when both terms in the product are of that form.

(3.5) BRANCHING RULE METHOD KRONECKER PRODUCTS FOR SO_{2k+1} AND Sp_{2k}

The branching rule method may be usefully applied to the other classical groups, SO_{2k+1} and Sp_{2k}, as well.

The group SO_{2k+1} in particular deserves attention due to the overcounting problem that occurs in the unasterisked version of (3.15b). It is unlikely, however, that any improvements could be made to (3.7) or (3.15a). Neither is it likely that, in Sp_{2k}, a simpler formula than (3.4) could be obtained. Nevertheless, for completeness, all the necessary branching rule method products for those groups are derived here.

Once again U_k seems the logical choice of subgroup due to the ease with which Kronecker products can be performed there. Unlike the situation for SO_{2k}, all the required branching rules for SO_{2k+1} and Sp_{2k} to U_k have been already produced (King 1975b). These are

$$SO_{2k+1}\downarrow U_k \qquad [\lambda]\downarrow\sum_{\zeta} \{\bar{\zeta};\lambda/\zeta F\} \tag{3.57a}$$

$$[\Delta;\lambda]\downarrow\sum_{\zeta}\{\bar{\zeta};(\lambda/\zeta F).Q\} \tag{3.57b}$$

$$Sp_{2k}\downarrow U_k \qquad \langle\lambda\rangle\downarrow\sum_{\zeta}\{\bar{\zeta};\lambda/\zeta D\} \tag{3.58}$$

The $\delta_H-\delta_G$ terms are obtained from table (3.1), and are

$$SO_{2k+1}\downarrow U_k \qquad \delta_G-\delta_H = (\tfrac{1}{2},\tfrac{1}{2}, \ldots , \tfrac{1}{2}) \tag{3.59a}$$

$$Sp_{2k}\downarrow U_k \qquad \delta_G-\delta_H = (1,1, \ldots , 1), \tag{3.59b}$$

where there are k terms in each vector.

Addition of such terms to irrep labels in U_k is equivalent to multiplication of those irreps by $\varepsilon^{\frac{1}{2}}$ and ε

for SO_{2k+1} and Sp_{2k} respectively. Thus it can be seen that the addition and subsequent subtraction of such terms in the manner prescribed by (3.26) is a null operation in U_k and so there is no reason to include such an operation; the formula (3.50), originally derived for the SO_{2k} products, also suffices for SO_{2k+1} and SO_{2k}.

Analogous to (3.54), the product formulae can be stated as

$$SO_{2k+1} \qquad [\lambda] \times [\mu] = \sum_{\zeta} [\{\bar{\zeta};\lambda/\zeta F\} \times \{\mu\}] \tag{3.60a}$$

$$[\lambda] \times [\Delta;\lambda] = \sum_{\zeta} [\varepsilon^{\frac{1}{2}} \times \{\bar{\zeta};\lambda/\zeta F\} \times \{\mu\}] \tag{3.60b}$$

$$[\Delta;\lambda] \times [\Delta;\lambda] = \sum_{\zeta} \{\bar{\zeta};(\lambda/\zeta.F).Q\}\cdot\{\mu\}] \tag{3.60c}$$

$$Sp_{2k} \qquad \langle\lambda\rangle \times \langle\mu\rangle = \sum_{\zeta} \langle\{\bar{\zeta};\lambda/\zeta D\} \times \{\mu\}\rangle \quad . \tag{3.61}$$

as before, the explicit reference to U_k may be removed by using (3.2) and (3.28). This gives

$$SO_{2k+1} \qquad [\lambda] \times [\mu] = \sum_{\zeta,\eta} [\bar{\zeta};(\lambda/\zeta.\eta.F).(\mu/\eta)] \tag{3.62a}$$

$$[\lambda] \times [\Delta;\mu] = \sum_{\zeta,\eta} [\Delta;\bar{\zeta};(\lambda/\zeta.\eta.F).(\mu/\eta)] \tag{3.62b}$$

$$[\Delta;\lambda] \times [\Delta;\mu] = \sum_{\zeta,\eta} [\bar{\zeta};(\lambda/\zeta.\eta.F).(\mu/\eta).Q] \tag{3.62c}$$

$$Sp_{2k} \qquad \langle\lambda\rangle \times \langle\mu\rangle = \sum_{\zeta,\eta} \langle\bar{\zeta};(\lambda/\zeta.\eta.D).(\mu/\eta)\rangle \tag{3.63}$$

All the rules (3.60) - (3.63) must be modified so that the labels are firstly U_k-standard using the rules of table (2.6) and secondly standard in either SO_{2k+1} or Sp_{2k}, using the rules of table (2.8).

Comparison of these formulae with their equivalents in section (3.1), shows that equations (3.60c) and (3.62c) offer considerable improvement on (3.15b). The difference is entirely due to the fact that the U_k modification rule provides a much more efficient cut-off than does the appropriate SO_{2k+1} rule.

As an illustration of this fact, consider the product $[\Delta;1] \times [\Delta;1]$ in SO_9. Equation (3.15b) gives

$$[\Delta;1] \times [\Delta;1] = \sum_{\zeta} \tfrac{1}{2}[Q.(1/\zeta).(1/\zeta)]$$

$$= \tfrac{1}{2}[Q.(1/0).(1/0)] + \tfrac{1}{2}[Q.0]$$

$$= \tfrac{1}{2}([Q.2] + [Q.1^2] + [Q])$$

Equation (3.16) indicates that terms in the Q series of up to eleven parts must be considered in the first term, ten in the second and nine in the third. Thus

$$\begin{aligned}\tfrac{1}{2}([Q.2]+[Q.1^2]+[Q]) = \tfrac{1}{2}(&[31^{10}]+[31^9] + [31^8] + [31^7] \\ &+[31^6] + [31^5] + [31^4] + [31^3] +[31^2] + [31] + [3] \\ &+[2^21^8] + [2^21^7] + [2^21^6] + [2^21^5] + [2^21^4] \\ &+[2^21^3] + [2^21^2] + [2^21] + [2^2] + [21^{11}] + 2[21^{10}] \\ &+2[21^9] + 2[21^8] + 2[21^7] + 2[21^6] + 2[21^5] + 2[21^4]\end{aligned}$$

$$+ 2[21^3] + 2[21^2] + 2[21] + [2] + [1^{12}] + [1^{11}]$$

$$+ 2[1^{10}] + 2[1^9] + 2[1^8] + 2[1^7] + 2[1^6] + 2[1^5]$$

$$+ 2[1^4] + 2[1^3] + 2[1^2] + [1] + [0]) \quad .$$

Applying the appropriate modification rule from table (2.3) leads to

$$[\Delta;1] \times [\Delta;1] = \tfrac{1}{2}(2[31^3] + 2[31^2] + 2[31] + 2[3]$$

$$+ 2[2^21^2] + 2[2^21] + 2[2^2] + 4[21^3]$$

$$+ 4[21^2] + 4[21] + 2[2] + 4[1^4]$$

$$+ 4[1^3] + 4[1^2] + 2[1] + 2[0]) \quad .$$

Alternatively, applying (3.60c) to the product means first finding $\sum_\zeta\{\bar{\zeta};(\lambda/\zeta F).Q\}$ in U_4. This is given by

$$\sum_\zeta \{\bar{\zeta};(1/\zeta F).Q\} = \{\bar{1};Q\} + \{Q\} + \{1.Q\} \quad .$$

The U_4 modification rules restrict the terms in the Q series to five in the first term and four in the second and third terms. Thus the branching rule gives

$$\{\bar{1};0\} + \{\bar{1};1\} + \{\bar{1};1^2\} + \{\bar{1};1^3\} + \{\ 0\ \} + 2\{1\}$$

$$+ 2\{1^2\} + 2\{1^3\} + \{1^4\} + \{2\} + \{21\} + \{21^2\} + \{21^3\}$$

Multiplying this by $\{1\}$ in U_4 gives

$$\{31^3\} + \{31^2\} + \{31\} + \{3\} + \{2^21^2\} + \{2^21\} + \{2^2\}$$

$$+2\{21^3\} + \{\bar{1};21^2\} + 3\{21^2\} + \{\bar{1};21\} + 3\{21\}$$

$$+ \{\bar{1};2\} + 2\{2\} + 2\{1^4\} + \{\bar{1};1^3\} + 3\{1^3\} + \{\bar{1};1^2\}$$

$$+ 3\{1^2\} + \{\bar{1};1\} + 2\{1\} + \{0\}.$$

This sum must be modified by the SO_{2k+1} tensor modification rule (table 2.8) to give the final result

$$[\Delta;1] \times [\Delta;1] = [31^3] + [31^2] + [31] + [3] + [2^21^2]$$

$$+ [2^21] + [2^2] + 2[21^3] + 2[21^2] + 2[21] + 2[2]$$

$$+ 2[1^4] + 2[1^3] + 2[1^2] + [1] + [0].$$

Clearly the latter method is the most efficient in the number of intermediate terms.

None of the other formulae for SO_{2k+1} or Sp_{2k} offer any improvement on those of section (3.1).

(3.6) KRONECKER PRODUCTS FOR THE EXCEPTIONAL GROUPS

While the branching rule method cannot offer neat Kronecker product formulae for the exceptional groups, as it did for the classical groups, it does offer a considerable increase in efficiency over any previous methods. In fact the method was developed primarily with the exceptional groups in mind (King 1981a).

The extent to which the branching rule method may be used depends very much on the degree to which the appropriat branching rules are known. Except for the very simplest case, the rule $G_2 \downarrow SU_3$, there are no general formulae known for branching rules involving exceptional groups. However that is not to say that the branching rules are not known, indeed extensive tables are available in the literature (Wybourne and Bowick 1977, Wybourne 1979, McKay and Patera 1980). Given that the branching rule method involves a very asymmetric formula (3.26), these rules should suffice for most purposes.

The logical choice of subgroups for the products are the subgroups being used to label the irreps. Besides providing the simplest branching rules, with one exception, Kronecker products are easy to evaluate in these groups. The exception is the subgroup of F_4, SO_9, and more will be said of this later.

Since the branching rules are available the next thing to consider is the question of the $\underset{\sim}{\delta}_G - \underset{\sim}{\delta}_H$ terms to be employed in the implementation of (3.26). From table (3.1), for the subgroups chosen, it can be seen that the $\underset{\sim}{\delta}_G - \underset{\sim}{\delta}_H$ are non-zero only in the first term. These are presented in table (3.2). Unfortunately the simplifications that allowed the classical group products to be performed using (3.50) do not occur using these subgroups and hence, in general, the $\underset{\sim}{\delta}_G - \underset{\sim}{\delta}_H$ terms must be added and subtracted in the prescribed manner.

The modification rules to make the final product standard in the appropriate exceptional group are those of table (2.7b). In general these involve the addition of a

TABLE (3.2): $\delta_G - \delta_H$ for the exceptional groups

Group	Subgroup	$\underset{\sim}{\delta}_G - \underset{\sim}{\delta}_H$
G_2	SU_3	(1,0)
F_4	SO_9	(2,0,0,0)
E_6	$SU_2 x SU_6$	(10,0,0,0,0,0)
E_7	SU_8	(10,0,0,0,0,0,0)
E_8	SU_9	(21,0,0,0,0,0,0)

particular constant to some parts of a label, and are to be applied repeatedly until either a standard label is produced or the character is shown to vanish.

The products of G_2 may be written out explicitly as the branching rule $G_2 \downarrow SU_3$ may be given as (Fronsdal 1962, King and Al-Qubanchi 1978)

$$G_2 \downarrow SU_3 \qquad (\lambda_1,\lambda_2) \downarrow \sum_{r=0}^{\lambda_2} \sum_{s=0}^{\lambda_1-2\lambda_2} \sum_{t=0}^{r+s} \{\lambda_1-\lambda_2+r-t, \lambda_2+s-t\} \ , \qquad (3.64)$$

The product rule is therefore

$$G_2 \qquad (\lambda_1,\lambda_2) \times (\mu_1,\mu_2) = \sum_{r=0}^{\lambda_2} \sum_{s=0}^{\lambda_1-2\lambda_1} \sum_{t=0}^{r+s} (\{\lambda_1-\lambda_2+r-t, \lambda_2+s-t\}$$

$$\times \ \{\mu_1+1, \mu_2\} - (\underset{\sim}{1},\underset{\sim}{0})) \ , \qquad (3.65)$$

where the internal product is performed in SU_3. As an example consider the product (21) x (1). The branching rule gives

$$G_2 \downarrow SU_3 \qquad (1) \downarrow \{1^2\} + \{1\} + \{0\} \quad ,$$

which when multiplied by $\{2+1,1\} = \{31\}$ in SU_3 produces

$$SU_3 \qquad (\{1^2\} + \{1\} + \{0\}) \times \{31\} = \{42\} + \{41\} + \{32\} + \{31\}$$

$$+ \{3\} + \{21\} + \{2\} \ .$$

After subtraction of $(\underset{\sim}{1},\underset{\sim}{0})$ this becomes

G_2 (32) + (31) + (2^2) + (21) + (2) + (1^2) + (1),

which must be modified via table (2.7b), since (32) (2^2) and (1^2) are non standard and are equivalent to 0, -(21) and 0 respectively, to give as the final answer.

$$(21) \times (1) = (31) + (2) + (1)$$

While the branching rules for the other exceptional groups cannot be written in general form, the procedure to obtain their products is exactly the same as for the G_2 products. For the F_4 products, however, ease of calculation may make U_4 a more suitable subgroup than SO_9. In this case the $SO_9 \downarrow U_4$ branching rules (3.57) can be chained with the $F_4 \downarrow SO_9$ rules read from a table. The $\underset{\sim}{\delta}_{F_4} - \underset{\sim}{\delta}_{U_4}$ term is $(\underset{\sim}{\tfrac{5}{2}}\ \underset{\sim}{\tfrac{1}{2}}\ \underset{\sim}{\tfrac{1}{2}}\ \underset{\sim}{\tfrac{1}{2}})$ but the arguments given in section (3.5) concerning columns in products using U_k as a subgroup apply here as well, and so adding and subtracting a term of the form $(\underset{\sim}{2}\ \underset{\sim}{0}\ \underset{\sim}{0}\ \underset{\sim}{0})$ will suffice. In E_6 products the subgroup is $SU_2 \times SU_6$, a direct product group. Kronecker products in such groups, H and J say, are given by the formula

$$H \times J \qquad (\underset{\sim}{\lambda}_H \times \underset{\sim}{\mu}_J) \times (\nu_H \times \kappa_J)$$

$$= \sum_{\rho_H} \sum_{\sigma_J} K_{\lambda_H \nu_H}^{\rho_H} K_{\mu_J \kappa_J}^{\sigma_J} (\rho_H \times \sigma_J) \qquad (3.66)$$

The addition of the $\underset{\sim}{\delta}_G - \underset{\sim}{\delta}_H$ terms sometimes means that high weight products have to be considered in the subgroup. This is especially a problem in the higher rank E-series of groups, and in those groups it has been found possible to replace $\underset{\sim}{\delta}_G-\underset{\sim}{\delta}_H$ with a smaller term, $\underset{\sim}{\delta}$, in some instances. In E_6 it has been possible to show, noting the trivial nature of products in SU_2, that the term $\underset{\sim}{\delta}_G-\underset{\sim}{\delta}_H$ may be dispensed with altogether if the product is ordered appropriately. Specifically the minimum value of $\underset{\sim}{\delta}$ for the product $(s:\lambda) \times (t:\mu)$ is given by

$$\underset{\sim}{\delta} = \begin{cases} (0,0,0,0,0,0) & \text{if } \; s-t \leqslant 0 \\ (s-t,0,0,0,0,0) & \text{if } \; 0 < s-t \leqslant 10 \\ (10,0,0,0,0,0) & \text{if } \; s-t > 10 \end{cases} \tag{3.67}$$

For E_7 and E_8 the minimum value of $\underset{\sim}{\delta}$ in the product $(\lambda) \times (\mu)$ is

$$\underset{\sim}{\delta} = \begin{cases} (0,0, \; \ldots \; , 0) & \text{if } \; \lambda_1-\mu_1+\mu_2 \leqslant 0 \\ (\lambda_1-\mu_1+\mu_2,0,0 \; \ldots \; , 0) & \text{if } \; 0 < \lambda_1-\mu_1+\mu_2 \leqslant d \\ (d,0,0, \; \ldots \; , 0) & \text{if } \; \lambda_1-\mu_1+\mu_2 > d \; , \end{cases} \tag{3.68}$$

where d is 10 for E_7 and 21 for E_8. For example, in the E_8 product $(21) \times (21)$, $\underset{\sim}{\delta}$ is $(1,0,0,0,0,0,0,0)$ instead of the more unwieldly $\underset{\sim}{\delta}_{E_8} - \underset{\sim}{\delta}_{SU_9} = (21,0,0,0,0,0,0,0)$. However ordering the product to minimise $\underset{\sim}{\delta}$ may not always provide maximum efficiency. Maximum efficiency is almost always obtained by branching down the term with the smallest number of terms in the branching rule. For instance, in the E_8 product $(41^5) \times (21^7)$ the $\underset{\sim}{\delta}$ term may be neglected when the (41^5) irrep is branched down. However this means that 58 SU_9 products, many of high weight, have to be evaluated. In contrast, if the (21^7) irrep is branched down, while the leading term in $\underset{\sim}{\delta}$ is 3, there are only three SU_9 products to be evaluated.

CHAPTER 4

COMPUTING CONSIDERATIONS

The methods for evaluating Kronecker products outlined in chapter three allow the calculation by hand of many products previously calculable only by computer. This fact, however, is more of mathematical than physical importance. The physicist is primarily interested in whether products have been calculated at all and is only peripherally, if at all, interested in the elegance of the calculation methods. Extensive tables exist of a large proportion of the Kronecker products, dimensions, invariants, branching rules etc. that physicists require at present (for example Wybourne 1970, Slansky 1981, McKay and Patera 1980). In a sense such tables only need to be produced once.

Yet character methods have an importance beyond their ability to produce tables of properties that physicists may currently need. This importance lies in their flexibility of use, primarily through the algebra of S-functions. New tasks require new methods and algorithms, and character methods often offer the most efficient means of producing new results.

Part of the motivation for the development of the methods in this thesis has been the desire to systematize the situation regarding the Kronecker products so that a complete set of computer algorithms could be produced. Such a program was not to be seen firstly as a 'table producer', although it would have to have such a capacity, but as a research tool to aid in the development and checking of new character equations. To be useful the program would have to be reasonably fast at basic operations such as the Littlewood-Richardson rule so that large problems involving series can be made tractable.

Such a program has been under development at the University of Canterbury, originally using a Burroughs B6700 machine, but mostly on a Prime 750 computer. The language chosen was PASCAL due to the portability and power of this language. This choice has had an enormous effect on the structure and data representation concepts of the program which has been designated SCHUR.

(4.1) DATA REPRESENTATION

The numeric nature of partitions makes them ideal group theoretic labels for use in computers. This is especially so since the S-function methods involve only direct manipulations of these labels. Partitions can be easily stored in a computer as a one dimensional array of integers with an extra integer to give a multiplicity index for the partition. It may be thought that using integer arrays for this purpose is expensive in terms of memory used in comparison with more sophisticated storage techniques, however the number of partitions involved in most tasks SCHUR is involved in is not very large when viewed from a computing perspective. An advantage of using small integer arrays is the speed of access to the elements when compared with other methods such as character strings.

The two basic S-function operations of outer multiplication and S-function division are not difficult to computerise given the combinatoric nature of the methods to evaluate them, however difficulties arise in the storage of results. In general these operations, and in fact most S-function operations, produce a sum of S-functions as a result. One approach to the storage of sums of S-functions, and one taken earlier by Butler (1970, unpublished) in his FORTRAN program SFN, is to use two dimensional arrays. There are many disadvantages in following such an approach. Insertion and deletion of partitions from a large list is very unweildly and this makes a frequently necessary operation like sorting a list extremely involved, and hence slow. A far more flexable approach is found in the use of 'linked lists', where each data point, in this case an S-function or group irrep label, also contains the address in memory of the next data point in the sum. This provides for each insertion and deletion of data points and hence fast sorting procedures. PASCAL is well suited to this sort of data structure which can be most effectively programmed using the PASCAL data types 'record' and 'pointer'. Thus an S-function, in SCHUR, is represented by a record containing a 1-dimensional array, for the partition, an integer, for the multiplicity, and a pointer, for the address of the next S-function. A group irrep requires a boolean to indicate whether a spinor index is present, an extra label for special purposes, such as the '+' in SO_{2k}, and an optional (by the use of variant records) extra partition for irreps with double partition labels.

The greatest affect this data stucture has on the underlying concepts of the program is that the data points themselves no longer have names. Only sums of S-functions and irreps have names.

(4.2) PROGRAM STRUCTURE AND SCOPE

The chosen 'linked list' data structure has the effect of homogenizing the input and output of all procedures. All the S-function operation procedures for instance have S-function lists as both input and output. This fact gives the program great flexibility in that the same procedures can be linked together in different ways, building up the multitude of formulae in chapter three.

At the lowest level in the program are the two basic S-function operations. These in turn produce other S-function operations specifically the inner product of S-functions (i.e. Kronecker products of S_n) and the plethysm of S-functions (Littlewood 1940, p206). These two operations are treated differently from any others in the program because only recursive formulae are known for them. Several procedures of SCHUR are designed to produce tables of these operations and store them on file for later use. The inner products are stored in n-independent form (Murnaghan 1938). The operation of plethysm, conventionally denoted '$\otimes$', is especially difficult to program because, while it is distributive with respect to addition on the right (4.1) there are difficulties involved with having a sum (or difference) of S-functions on the left (4.2).

$$\{\lambda\} \otimes (\{\mu\} \pm \{\nu\}) = (\{\lambda\} \otimes \{\mu\}) \pm (\{\lambda\} \otimes \{\nu\}) \tag{4.1}$$

$$(\{\lambda\} + \{\mu\}) \otimes \{\nu\} = \sum_{\sigma} (\{\lambda\} \otimes \{\nu/\sigma\}) \cdot (\{\mu\} \otimes \{\sigma\}) \tag{4.2a}$$

$$(\{\lambda\} - \{\mu\}) \otimes \{\nu\} = \sum_{\sigma} (-1)^{s} (\{\lambda\} \times \{\nu/\sigma\}) \cdot (\{\mu\} \otimes \{\tilde{\sigma}\}) \tag{4.2b}$$

However, a procedure which evaluates plethysms of the form $(\{\lambda\} \pm \{\mu\} \pm \cdots \pm \{\nu\}) \otimes (\{\alpha\} \pm \{\beta\} \pm \cdots \pm \{\gamma\})$ is available within SCHUR.

Along with the basic S-function operations, at the lowest level of the program (in the sense that the procedures are self-contained) are the S-function series procedures and the dimensional package. The series procedures produce any of the series of chapter 2 (and the two in section 3.3) up to any arbitrary cut-off provided by a partition. The dimensional package gives dimensions of irreps in the groups so far mentioned, including the symmetric group, S_n. These procedures use the formulae of table (2.4) and those of El Samra and King (1979). Besides being of interest in themselves the dimensions of irreps provide a useful check on the Kronecker products since

$$D_{\lambda} D_{\mu} = \sum_{\nu} K^{\nu}_{\lambda\mu} D_{\nu} \tag{4.3}$$

where D is the dimension and K is the Kronecker product multiplicity in some group. Dimensions do not provide a complete check but do place severe restrictions on any errors that may occur.

The Kronecker product procedures are of two types; those calculated by the methods of section (3.1) and those calculated by the branching rule method. The former are the products in U_n, SU_n, O_n, Sp_{2k} and SO_{2k+1}. Of the latter the products of SO_{2k} rely on the products of U_k while the products of the exceptional groups, G_2, F_4, E_6, E_7 and E_8, at this stage in the development of SCHUR, require physical input of the branching rules to SU_3, SO_9, $SU_2 \times SU_6$, SU_8 and SU_9 respectively. A compact formula exists for the branching $G_2 \downarrow SU_3$ (Eqn. (3.64) King and Al-Qubanchi, 1978) so this will be programmed, however as no such formula exists for the other exceptional group branchings, these branchings will be placed on a look-up table for the programs use.

Another important use of the program is in the evaluation of branching rules. The procedures for individual branching rules may be quickly programmed as required using the S-function operation and series procedures. King (1975b) gives an extremely useful review of the methods and formulae involved in the branching rules of the classical groups. Two examples of the use of SCHUR to evaluate branching rules are given in appendix II and appendix III. In appendix II there is a table of the branching $SO_{10} \downarrow U_1 \times SU_5$, while in appendix III the branching $SU_5 \downarrow U_1 \times SU_2 \times SU_3$ is evaluated. The former branching uses the formula of section (3.3) whilst the latter uses a formula due to King (1975b).

$$SU_{p+q} \downarrow \sum_{\zeta} \{\lambda/\zeta\} \times \{\zeta\} , \tag{4.4}$$

given suitable modifications. The U_1 labels are obtained using the results of King (1981b). These groups are of interest in grand unified theories.

(4.3) THE INTERACTIVE MODE

The usefulness of a program is, to a large extent, dependent upon the top level of the program, the way the program appears to the person using it. In view of this a provisional top level of SCHUR was written to allow simple evaluation of Kronecker products, S-function operations and dimensions of irreps. In addition a package for obtaining symmetrised Kronecker powers for the classical groups was partially implemented. An example of a session with SCHUR is given below. While a thesis is hardly the place for a program manual, some of the command structure needs comment. The program is organised on several levels. At the top level, indicated by the prompt '#?', entrance to lower levels, where any calculation is done, is gained by use of the commands 'products', 'basics', 'dimensions' or 'powers'. These commands lead to the Kronecker product, S-function, dimension and symmetrised Kronecker powers packages respectively. After selection of a package the program will ask for selection of a group, or an operation. After this the program will give a prompt, depending on the package, asking for the input of partitions or sums of partitions after which calculation will ensue. Because of printer limitations the program is unable to place a bar above contravariant partitions and also uses an 's' for the basic spin irrep, 'Δ'. User input is underlined.

```
OK, seg schur
SCHUR_________________9/11/81
#?
products
KRONECKER PRODUCTS [ 9 / 11 / 81 ]
GROUP?
su 3
GROUP IS su(3)
]?
2 1  2 1
{21} x {21} =
{42} + {33} + {3} + 2{21} + {0}
]?
1 ; 1  2 1
{1;1} x {21} =
{42} + {33} + {3} + 2{21} + {0}
]?
stop
GROUP?
u 3
GROUP IS u(3)
]?
1 ; 1  2 1
{1;1} x {21} =
{1;31} + {3} + {1;22} + 2{21} + {111}
]?
stop
GROUP?
u 6
GROUP IS u(6)
]?
4 3 + 1 1   3 3 + 2
{11} + {43} x {2} + {33} =
{76} + {751} + {742} + {733} + {661} + {652}
 + {6511} + {643} + {6421} + {6331} + {63} + {5521}
 + {5431} + {5422} + {54} + {5332} + {531} + {4432}
 + {441} + {44} + {4333} + {432} + {431} + {3311}
 + {31} + {211}
]?
1 1 ; 2 + 2 1   2 1 1 1 1 1 + 2 ; 0
{21} + {11;2} x {2;0} + {211111} =
{421111} + {4111} + {331111} + {322111} + {3211} + {31111}
 + {2;21} + {31;2} + {211;2} + {1;2} + {1;11} + {21;1}
 + {111;1} + {1} + {11;0}
]?
stop
GROUP?
o 6
GROUP IS o(6)
]?
2 1 1   3 2
[211] x [32] =
[531] + 2[52] + [511] + [441] + [432] + 4[43]
 + 4[421] + 4[41] + 3[331] + 2[322] + 6[32] + 4[311]
 + 2[3] + 3[221] + 4[21] + [111]
]?
(3)s1  2 + s2
3[s;1] x [s;2] + [2] =
3[411] + 6[41] + 6[4] + 3[321] + 6[32] + 6[311]
 + 12[31] + 6[3] + 3[s;3] + 3[221] + 6[22] + 6[211]
 + 12[21] + 3[s;21] + 6[2] + 3[s;2] + 3[111] + 6[11]
 + 3[s;11] + 6[1] + 3[s;1] + 3[s;0]
]?
stop
GROUP?
so 8
GROUP IS so(8)
]?
2 1 1 1+ + s1- + 2  s2 1-
[2] + [s;1]- + [2111]+ x [s;21]- =
[4211]- + [42] + [411] + [s;41]- + [3311]- + [33]
 + [3221]- + 3[321] + [s;32]- + 2[3111]- + [3111]+ + [s;311]-
 + 2[31] + [s;31]+ + [s;3]- + [222] + 2[2211]- + [2211]+
 + [s;221]- + 2[22] + [s;22]+ + 3[211] + 2[s;211]+ + 2[s;21]-
 + [2] + 2[s;2]+ + [1111]- + [s;1111]+ + [s;111]- + [11]
 + 2[s;11]+ + [s;1]-
]?
stop
GROUP?
sp 6
GROUP IS sp(6)
]?
2 1 + 3 1 1  2 + 1 1
<311> + <21> x <11> + <2> =
<511> + 2<421> + 3<41> + <331> + <322> + 4<32>
 + 5<311> + 3<3> + 4<221> + 6<21> + 3<111> + 2<1>
]?
stop
GROUP?
s 5
GROUP IS s(5)
LOADING TABLE OF PRODUCTS
]?
3 2  2 1 1 1
{32} x {2111} =
{32} + {311} + {221} + {2111}
]?
4 1 + 1 1 1 1  3 2
{1111} + {41} x {32} =
{1111} NOT VALID IN S(5)
]?
4 1 + 1 1 1 1 1  3 2
{11111} + {41} x {32} =
{41} + {32} + {311} + 2{221}
]?
```

```
st
GROUP?
e 8
GROUP IS e(8)
FIRST TERM?
2 1 1 1 1 1 1 1
BRANCHING?
2 1 1 1 1 1 1 1 + 1 1 1 1 1 1 + 1 1 1
SECOND TERM?
2 1 1 1 1 1 1 1
(21111111) x (21111111) =
(42222222) + (3111111) + (21111111) + (21) + (0)
SECOND TERM?
2 1
(21111111) x (21) =
(42111111) + (3111111) + (3) + (21111111) + (21)
SECOND TERM?
3
(21111111) x (3) =
(51111111) + (42111111) + (411111) + (411) + (3111111) + (3)
 + (21)
SECOND TERM?
change
FIRST TERM?
2 1
BRANCHING?
2 1 + 2 1 ; 0 + 1 1 ; 1 1 + 1 ; 1 1 1 1 + 1 1 1 1 ; 1 + 1 ; 1
SECOND TERM?
2 1 1 1 1 1 1 1
(21) x (21111111) =
(42111111) + (3111111) + (3) + (21111111) + (21)
SECOND TERM?
2 1
(21) x (21) =
(42222222) + (42111111) + (42) + (411111) + (411) + (3111111)
 + (3) + (21111111) + (21) + (0)
SECOND TERM?
change
FIRST TERM?
2 1 1 1 1 1 1 1
BRANCHING?
2 1 1 1 1 1 1 1 + 1 1 1 + 1 1 1 ; 0
SECOND TERM?
2 1 + 3
(21111111) x (3) + (21) =
(51111111) + 2(42111111) + (411111) + (411) + 2(3111111) + 2(3)
 + (21111111) + 2(21)
SECOND TERM?
stop
GROUP?
e 7
GROUP IS e(7)
FIRST TERM?
```

```
2 1 1
BRANCHING?
2 1 1 + 2 1 1 ; 0 + 1 1 ; 1 1 + 1 ; 1
SECOND TERM?
2 1 1 1 1 1 1
(211) x (2111111) =
(4221111) + (311111) + (31) + (22) + (2111111) + (211)
SECOND TERM?
change
FIRST TERM?
2 1 1 + 1 1
BRANCHING?
1 1 1 1 1 1 + 1 1 + 2 1 1 + 2 2 2 2 2 1 1 +%
2 2 1 1 1 1 + 2 1 1 1 1 1 1
SECOND TERM?
1 1
(11) + (211) x (11) =
(3211111) + (321) + (3111) + (22) + (2111111) + (211)
 + (2) + (11) + (0)
SECOND TERM?
change
FIRST TERM?
1 1
BRANCHING?
1 1 1 1 1 1 + 1 1
SECOND TERM?
2 1 1 + 1 1
(11) x (11) + (211) =
(3211111) + (321) + (3111) + (22) + (2111111) + (211)
 + (2) + (11) + (0)
SECOND TERM?
stop
GROUP?
exit
#?
dimen
DIMENSIONS
GROUP?
u 9
GROUP IS u(9)
&?
2 2 1 1
DIMENSION OF {2211} IS 3402
&?
stop
GROUP?
o 6
GROUP IS o(6)
&?
2 1 1
DIMENSION OF [211] IS 90
&?
3 2
```

```
DIMENSION OF [32] IS 300
&?
stop
GROUP?
sp 6
GROUP IS sp(6)
&?
2 1 1
DIMENSION OF <211> IS 70
&?
3 2
DIMENSION OF <32> IS 350
&?
stop
GROUP?
exit
#?
basic
BASIC S-FN OPERATIONS
OPERATION?
outer
$?
2 1  2 1
{21} . {21} =
{42} + {411} + {33} + 2{321} + {3111} + {222}
 + {2211}
$?
4 3 + 1 1  3 3 + 2
{11} + {43} . {2} + {33} =
{76} + {751} + {742} + {733} + {661} + {652}
 + {6511} + {643} + {6421} + {6331} + {5521} + {5431}
 + {5422} + {5332} + {4432} + {4333} + {63} + {54}
 + {531} + {441} + {432} + {44} + {431} + {3311}
 + {31} + {211}
$?
stop
OPERATION?
skew
$?
3 2 2  2 1
{322} / {21} =
{31} + {22} + {211}
$?
5 4 3 2  3 1
{5432} / {31} =
{541} + 2{532} + {5311} + {5221} + 2{442} + {4411}
 + 2{433} + 3{4321} + {4222} + {3331} + {3322}
$?
stop
OPERATION?
inner
$?
6 1  5 2
```

```
{61} o {52} =
{61} + {52} + {511} + {43} + {421}
$?
5 1  4 2
{51} o {42} =
{51} + {42} + {411} + {33} + {321}
$?
4 1  3 2
{41} o {32} =
{41} + {32} + {311} + {221}
$?
3 1  2 2
{31} o {22} =
{211} + {31}
$?
stop
OPERATION?
rinner
$?
1  2
<1> o <2> =
<1> + <11> + <2> + <21> + <3>
$?
stop
OPERATION?
plethysm
$?
2  2
LOADING PLETHYSMS
{2} X {2} =
{4} + {22}
$?
3 1  2 1
{31} X {21} =
{921} + {84} + 2{831} + {822} + 2{8211} + {81111}
 + {75} + 3{741} + 4{732} + 4{7311} + 3{7221} + 2{72111}
 + 3{651} + 5{642} + 5{6411} + 2{633} + 7{6321} + 3{63111}
 + {6222} + 2{62211} + {621111} + 2{552} + 2{5511} + 4{543}
 + 6{5421} + 3{54111} + 4{5331} + 3{5322} + 4{53211} + {531111}
 + {52221} + 3{4431} + 2{4422} + 2{44211} + 2{4332} + 2{43311}
 + {43221} + {432111} + {33321}
$?
4 3  2
{43} X {2} =
{86} + {842} + {761} + {752} + {7511} + {743}
 + {7331} + {662} + {6521} + {644} + {6431} + {6422}
 + {5531} + {5432} + {5333} + {4442}
$?
stop
OPERATION?
exit
#?
powers
```

```
SYMMETRISED POWERS
GROUP?
u 8
GROUP IS u(8)
%?
4  2 1
{4} X {21} = + {543} + {651} + {642} + {75} + {741} + {732}
 + 2{84} + {831} + {93} + {921} + {102} + {111}
%?
stop
GROUP?
o 8
GROUP IS o(8)
%?
2  2 1
[2] X {21} = + [51] + [42] + [321] + [4] + 2[31] + [22]
 + [211] + 2[2] + [11]
%?
stop
GROUP?
sp 8
GROUP IS sp(8)
%?
2  2 1
<2> X {21} = + <51> + <42> + <321> + <4> + 2<31> + <22>
 + <211> + 2<2> + <11>
%?
stop
GROUP?
ex
#?
end
OK.
```

(4.4) CURRENT DEVELOPMENTS AND FUTURE PROSPECTS

At present the flexibility of the program rests mainly in the PASCAL language, the pointer/list concept, and the power of S-function methods. It is relatively simple, and in many cases, trivial, to program a new formula using the basic procedures. However the process of learning the basics of the program so that modifications can be made is neither simple or convenient. In view of this, development of a 'workbench' mode for SCHUR will soon begin. In this mode a worker would be able to use much of the program not at present accessible in the interactive mode, to chain operations and modifications and hence obtain results for formulae not available intrinsically to the program. The dynamic storage methods used put this generalisation of the program within reach, greatly increasing the usefulness of the program to those interested in developing new formulae.

A second item, currently under development, is the expansion of the dimensions package to one that will produce more information about particular representations of groups. In particular the ease with which Kronecker products can be calculated within a group allows the eigenvalues of the generalised Casimir invariants $I_p^{\mu}(\lambda)$ (Englefield and King, 1980), for arbitrary power, p, to be evaluated using the formula of Okubo (1977). This would allow the calculation of several quantities of interest to physicists such as Dynkin indices, renormalisation group functions and anomalies.

Another project under consideration is the extension of the scope of the program from groups to supergroups.

Such an extension does not appear to be too difficult as similar methods to those used in this thesis may be used in supergroup context (Dondi and Jarvis 1981). Finally, while partition labels are clearly the most useful for the computer to work with, it may be that in some applications, other labels are more appropriate. It is, therefore, expedient that the program be able to communicate to the user in these labels. While it is impractical to use dimensional labels, as they are ambiguous, it is envisioned that an option will be added to allow input and output using Dynkin's (1957) labels to be accepted. This option would use King and Al-Qubanchi's (1981a) conversions.

PART II

CHAPTER 5

MULTIQUARK HADRONS

One of group theory's more fruitful applications to physics has been in quark theory. There are few other methods available to analyse such ephemeral particles as the unseen (unseeable?) quarks and gluons. In this chapter some basic properties of the theoretical exotic three-quark-three-antiquark ($q^3\bar{q}^3$) multiquark particles are evaluated using group theory and the largely phenomological concepts of the MIT bag model.

(5.1) INTRODUCTION

It now seems apparent that the hadrons are composite particles, made up of more fundamental particles called quarks. Classification of the baryons and mesons can be successfully performed using, at present, five varieties (called 'flavours') of quarks. In the model the quarks are combined in one of two canonical ways: quark-antiquark, for the mesons, and three quarks for the baryons. All other combinations of quarks are thought to be unphysical.

To explain the absence of the non-canonical groupings, an additional quantum number called 'colour' is introduced. Any of three different colours, usually labelled red, green and blue, may be assigned to a quark. The laws of colour combination, which vaguely follow the laws of primary colour addition, together with the rule that only colourless (i.e. white) combinations of quarks are observable, leads to the conclusion that the observable states must be of the form

$$q^m \bar{q}^n \quad , \qquad m = n \ (\mathrm{mod}\ 3). \tag{5.1}$$

The concept of colour can be usefully described using group theory. In this description the three colours span the defining representation $\{1\}^C$ of SU_3^C, called the colour group. The only observable states are those described by the representation $\{0\}^C$. A gauge theory of interactions can then be developed using SU_3^C. This theory is called quantum chromodynamics (QCD) and is very similar to the well established theory of QED which uses U_1 as a gauge group to describe electromagnetic interactions.

However successful QCD is, it is a very difficult theory to work with. In fact the difficulties are so acute that it cannot yet be proven whether QCD confines the coloured quarks inside their colourless hadrons. Because of the difficulties in using QCD, several 'QCD-inspired' models have been developed. One of these is the MIT bag model (Chodos et al. 1974, DeGrand et al. 1975). This model includes confinement of coloured combinations of quarks as an assumption, and uses only the very lowest order QCD interactions. It is not a fundamental theory since it has several parameters which must be fitted to obtain agreement with experiment. Of all the phenomological models, the MIT bag model has had the most success in reproducing properties, such as masses, of the known mesons and baryons.

It is obvious that the colour-singlet-only condition on observable states (5.1) allows other combinations of quarks beside $q\bar{q}$, q^3 and $\bar{q}^3$. For instance $q^2\bar{q}^2$, $q^4\bar{q}$, $q^8\bar{q}^2$ and $q^3\bar{q}^3$ are all admissable. These other states are commonly referred to as 'multiquark' states. However simple admissibility according to group theory is far from conclusive evidence for the existence of such states. Until the problem of making real calculations with QCD, or some successor to QCD, is solved, multiquark hadrons must only have the status of being interesting possibilities. At present the experimenta viability of such particles seems low.

In spite of all this, the nature of the MIT bag model makes calculations for multiquark hadrons almost as simple as those for the known hadrons. Indeed extensive bag model calculations have been made for multiquark states. Jaffe (1977) pioneered this work by examining $q^2\bar{q}^2$ and q^6 configurations. Workers at the University of Nijmegen (Aerts et al. 1980, Mulders et al. 1980, Aerts 1979, Mulders 1980) have studied the $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 configurations. Strottman (1978, 1979) has considered $q^4\bar{q}$, and even some $q^5\bar{q}^2$ configurations using more sophisticated group theoretical techniques. Bickerstaff (1980, 1982a) however, points out that there are some inconsistencies in his calculations. All these authors use simplifying approximations to evaluate the colour-hyperfine contribution to the mass. In practice this means that unnecessary degeneracies are present in the mass spectrum. With proper use of the available group theory Bickerstaff (1980, 1982b) was able to evaluate properties of the $q^2\bar{q}^2$, $q^4\bar{q}$ and q^6 configurations without the use of such simplifying approximations. The same methods (Bickerstaff and Wybourne 1980) will be followed here and applied to the particular configuration $q^3\bar{q}^3$.

A major consideration in the analysis of multiquark hadrons is the possibility of 'dissociation'. Dissociation is said to occur when a multiquark state splits up into two or more colour-neutral states whose total mass is less than that of the original multiquark state. This process is zeroth order in the quark-gluon coupling constant and hence does not require the creation or annihilation of any quarks or the mediation of any gluons. It has, therefore, been termed 'superallowed' by Jaffe (1977). The possibility of dissociation presents severe problems for the interpretation of bag model results. In the bag model all quarks and subgroups inside the particle are artificially confined by the bag's confinement mechanism. However, in a multiquark state, not all subgroups of quarks are coloured and so are confined for no good reason. All this has the effect of making the use of the bag model in multiquark state calculations somewhat unrealistic as it does not take into account any decay mode, least of all such an important one as dissociation (just 'falling apart'). Therefore the bag model is sometimes termed a 'zero-width' model, because it assumes infinite stability of its states.

The approach of Jaffe and Low (1979) takes all of the above into account and finds a relation between the bag-model-predicted states (now called 'primitives') and phase shifts in scattering experiments. Important in this approach is a dynamical quantity associated with the S-matrix called the P-matrix. While this sort of approach has had much success in interpreting phase shift data using the $q^2\bar{q}^2$ configuration (Jaffe 1977), its use in analysing $q^3\bar{q}^3$ data is strongly limited because of its inability to handle multibody (>2) decay channels. The main decay possibility for $q^3\bar{q}^3$ states is into three mesons.

All this notwithstanding it is conceivable that the bag model predictions for $q^3\bar{q}^3$ states may be of some use. The clearest application of these states would be in the interpretation of baryon-antibaryon scattering experiments. As more suitable accelerators have come into operation these experiments have become more common. In particular much interest has been shown in nucleon-antinucleon experiments (see say Dover and Richard 1979). Early experiments gave indications that several mesonic structures, often called barionium states, existed near the nucleon-antinucleon threshold energy (Caroll et al. 1974, Chaloupka et al. 1976, Bruckner et al. 1977). However later experiments raised considerable doubts about these states (see Tripp 1980 for a review, and Pietrzyk 1980 for some pertinent comments on the interpretations of barionium). This has caused many authors, notably Jaffe (1981), to abandon the concept of barionia altogether. Some workers still remain unsure (e.g. de Swart et al. 1981), however, and some new experiments still indicate the possible presence of at least one state (Sakamoto et al. 1982, Sai et al. 1982). Whilst final judgement should perhaps wait for results from LEAR, it seems most likely that the effects of multiquark states, if any exist, will be of a more subtle nature than the production of conventional particles or resonances.

The scope of this chapter is thus limited somewhat by the absence of a clear experimental situation and the limitations of the P-matrix formalism. After the next section, where the essentially group theoretical question of state classification is considered, the bag model eigenstates are evaluated. Following this, in section (5.4), the dissociation channels for the nucleon-antinucleon related states will be considered. Finally some brief comments and conclusions will be made.

(5.2) THE CLASSIFICATION OF MULTIQUARK STATES

Group theory provides much scope for alternate choices in the way a state is classified. The particular choices that are made in selecting a classification scheme must depend on which features of the states are desired to be made explicit. Much unnecessary work is created by the choice of an inappropriate classification scheme.

The classification scheme employed here will be that of Bickerstaff and Wybourne (1980). In this scheme the quantum states of a quark are seen to span the {1} irrep of U_{18}. Antisymmetry of fermionic states requires that a 3-quark state transforms under the representation $\{1^3\}$ of U_{18}. Similarly a 3-antiquark state transforms under the $\{\bar{1}^3\}$ irrep of U_{18}. The states can then be classified using the following group chain.

$$\begin{aligned} U_{18} &\downarrow [SU_{12} \downarrow SU_2^I \times (SU_6^{CS} \downarrow SU_2^S \times SU_3^C)] \\ &\times [U_6 \downarrow U_1^S \times (SU_6^{CS} \downarrow SU_2^S \times SU_3^C)] \\ &\downarrow SU_2^I \times U_1^S \times SU_2^S \times SU_3^C \quad , \end{aligned} \tag{2.1}$$

where the ordinary quarks ($O \equiv u,d$) transform under SU_{12} and the strange quarks under U_6. SU_2^I is the isospin group while U_1^S gives the strangeness quantum number. SU_6^{CS} is the colour-spin group which decomposes into $SU_2^S \times SU_3^C$ the product group formed by the spin group and the colour group. Implicit in this classification scheme is the restriction to three flavours u,d and s. Table (5.1) gives a list of all the necessary branching rules for this chain.

Several other classification schemes can be employed (Jaffe 1977a, Wybourne 1978). Most of these employ the flavour group, SU_3^{fl} or U_3^{fl}. However flavour group quantum numbers are strongly mixed in many situations and it is more profitable to ignore this group altogether. The key superiority of Bickerstaff and Wybourne's scheme over the others rests in its seperate treatment of ordinary quarks from strange quarks. As is explained in the next chapter

TABLE (5.1): Branching rules

(a) $U_{18} \downarrow SU_{12} \times U_6$

$\{1^3\} \downarrow \{1^3\}\{0\} + \{1^2\}\{1\} + \{1\}\{1^2\} + \{0\}\{1^3\}$

(b) $SU_{12} \downarrow SU_2^I \times SU_6^{CS}$

$\{0\} \downarrow \{0\}\{0\}$

$\{1\} \downarrow \{1\}\{1\}$

$\{1^2\} \downarrow \{2\}\{1^2\} + \{0\}\{2\}$

$\{1^3\} \downarrow \{3\}\{1^3\} + \{1\}\{21\}$

note: Isospin is half SU_2^I label

(c) $U_6 \downarrow U_1^S \times SU_6^{CS}$

$\{0\} \downarrow \{0\}\{0\}$

$\{1\} \downarrow \{1\}\{1\}$

$\{1^2\} \downarrow \{2\}\{1^2\}$

$\{1^3\} \downarrow \{3\}\{1^3\}$

note: Strangeness is negative the U_1^S label

(d) $SU_6^{CS} \downarrow SU_2^S \times SU_3^C$

$\{0\} \downarrow {}^1\{0\}$

$\{1\} \downarrow {}^2\{1\}$

$\{2\} \downarrow {}^1\{1^2\} + {}^3\{2\}$

$\{1^2\} \downarrow {}^3\{1^2\} + {}^1\{2\}$

$\{21\} \downarrow {}^2\{0\} + {}^2\{3\} + {}^2\{21\} + {}^4\{21\}$

the colour hyperfine interaction operator includes a tensor operator at the SU_6^{CS} level, and is also dependent on the mass of the quarks undergoing gluon exchange It is therefore important that the SU_6^{CS} content for each group of quarks of the same mass be explicit in the state labels. In practice, ease of calculation is improved if the $SU_2^S \times SU_3^C$ contents are also displayed explicitly. In this scheme antiquarks and quarks are coupled only at the lowest level, $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$. This has the advantage of keeping track of the number of strange quark-antiquark pairs. If the quarks and antiquarks are coupled at the U_{18} level using the composite partition notation of Abramsky and King (1970), there arise ambiguities between the states with and without the pairs. Thus in the meson sector there is ambiguity between the η and the η' in the U_{18} coupled model. In the $SU_2^I \times U_1^S \times SU_2^S \times SU_3^C$ coupled model there is clearly an η_o made up only of ordinary quarks and an η_s made only of strange quarks. While the U_{18} coupled scheme may be closer to reality in this, calculations are far easier where there is no ambiguity. Another advantage of lower level coupling is that the irreps of the various groups are of lower weight.

Remembering that the colour hypothesis demands that the only allowed states are those with $\{0\}^C$ as a final SU_3^C label, a typical basis state for the $q^3\bar{q}^3$ configuration can be written as

$$|(q_o^x I_o \lambda_o^{CS} S_o \mu_o^C, q_s^{3-x} \mathcal{S}_s \lambda_s^{CS} S_s \mu_s^C) S_{os} \mu_{os}^C$$

$$(q_{\bar{o}}^y I_{\bar{o}} \lambda_{\bar{o}}^{CS} S_{\bar{o}} \mu_{\bar{o}}^C, q_{\bar{s}}^{3-y} \mathcal{S}_{\bar{s}} \lambda_{\bar{s}}^{CS} S_{\bar{s}} \mu_{\bar{s}}^C) S_{\overline{os}} \mu_{\overline{os}}^C ; I \mathcal{S} S O^C \rangle \ . \qquad (5.2)$$

In this I is an isospin label, $\mathcal{S}$ is a strangeness label, λ^{CS} labels SU_6^{CS}, S is a spin label and μ^C labels SU_3^C. Clearly $\mu_{os}^C = \bar{\mu}_{\overline{os}}^C$. For historical reasons not all the labels used are partition labels. Spin and Isospin are commonly labelled by half the SU_2 partition label, however often spin is labelled by the dimension of the SU_2 label, which is just one more than the SU_2 label. Since SU_2^S occurs only in conjunction with SU_3^C, spin is denoted by the dimensional label as a left superscript to the colour label. The strangeness quantum number is just the negative of the expected U_1 label.

There are 590 $q^3\bar{q}^3$ states, 356 ignoring antiparticles, which is too many to profitably list in full. However to determine the underlying basis states of the bag model eigenstates given in table (5.4), use can be made of table (5.2) which gives $SU_{12} \downarrow SU_2^I \times SU_6^{CS}$ and $U_6 \downarrow U_1^{\mathcal{S}} \times SU_6^{CS}$ parentage of the basis states. Table (5.1d) can be used to establish the $SU_2^S \times SU_3^C$ labels. In tables (5.2) and (5.4) the script capitals refer to the colour-spin (SU_6^{CS}) content. In table (5.3) the especially interesting nucleon-antinucleon related states are explicitly listed.

TABLE (5.2): Parentage of states in table 5.4

Asterisked states have unlisted anti-states.

${}^{1,3,5,7}A$	$o^3\ \frac{3}{2}\ 1^3;\ \bar{o}^3\ \frac{3}{2}\ 1^3$	
	$s^3\ -3\ 1^3;\ \bar{o}^3\ \frac{3}{2}\ 1^3$	
	$s^3\ -3\ 1^3;\ \bar{s}^3\ 3\ 1^3$	
${}^{1,3,5}B$	$o^3\ \frac{3}{2}\ 1^3;\ \bar{o}^3\ \frac{1}{2}\ 2^4 1$	
	$s^2\ -3\ 1^3;\ \bar{o}^3\ \frac{1}{2}\ 2^4 1$	*
${}^{1,3,5,7}C$	$o^3\ \frac{1}{2}\ 21;\ \bar{o}^3\ \frac{1}{2}\ 2^4 1$	
${}^{1,3,5,7}D$	$o^2\ 1\ 1^2,\ s\ -1\ 1;\ \bar{o}^2\ 1\ 1^4, \bar{s}\ 1\ 1^5$	
	$s^2\ -1\ 1^2,\ o\ \frac{1}{2}\ 1;\ \bar{o}^2\ 1\ 1^4, \bar{s}\ 1\ 1^5$	*
	$s^2\ -2\ 1^2,\ o\ \frac{1}{2}\ 1;\ \bar{s}^2\ 2\ 1^4, \bar{o}\ \frac{1}{2}\ 1^5$	
${}^{1,3,5,7}E$	$o^2\ 0\ 2,\ s\ -1\ 1;\ \bar{o}^2\ 0\ 2^2,\ \bar{s}\ 1\ 1$	
${}^{1,3,5,7}F$	$o^2\ 1\ 1^2,\ s\ -1\ 1;\ \bar{o}^2\ 0\ 2^5,\ \bar{s}\ 1\ 1$	*
	$s^2 -2\ 1^2,\ o\ \frac{1}{2}\ 1;\ \bar{o}^2\ 0\ 2^5,\ \bar{s}\ 1\ 1$	*
${}^{1,3,5,7}G$	$o^2\ 1\ 1^2, s\ -1\ 1;\ \bar{o}^3\ \frac{3}{2}\ 1^3$	*
	$s^2\ -2\ 1^2, o\ \frac{1}{2}\ 1;\ \bar{o}^3\ \frac{3}{2}\ 1^3$	*
	$s^2\ -2\ 1^2, o\ \frac{1}{2}\ 1;\ \bar{s}^3\ 3\ 1^3$	*
	$1^2\ 1\ 1^2, s\ -1\ 1;\ \bar{s}^3\ 3\ 1^3$	*
${}^{1,3,5,7}H$	$o^2\ 1\ 1^2,\ s\ -1\ 1;\ \bar{o}^3\ \frac{1}{2}\ 2^4 1$	*
	$s^2\ -2\ 1^2,\ o\ \frac{1}{2}\ 1;\ \bar{o}^3\ \frac{1}{2}\ 2^4 1$	*

TABLE (5.2) Continued

$^{1,3,5}I$	o^2 0 2, s -1 1; $\bar{o}^3$ $\frac{3}{2}$ 1^3	*
	o^2 0 2, s -1 1; $\bar{s}^3$ 3 1^3	*
$^{1,3,5,7}J$	o^2 0 2, s -1 1; $\bar{o}^3$ $\frac{1}{2}$ $2^4 1$	

TABLE (5.3): Nucleon-antinucleon related states

(a) Spin 0 $\quad ^1C$ $o^3\bar{o}^3$ states

$|(q^3 I_q \lambda_q^{CS} S_q \mu_q^C ; \bar{q}^3 I_{\bar{q}} \lambda_{\bar{q}}^{CS} S_{\bar{q}} \mu_{\bar{q}}^C) I\ S\ O^C\rangle$

$^1C_3$	$\|(o^3\ \frac{1}{2}\ 21\ \ {}^20\ ;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^20)\frac{1}{2}\ {}^10\rangle$
$^1C_2$	$\|(o^3\ \frac{1}{2}\ 21\ \ {}^23\ ;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^23^2)\frac{1}{2}\ {}^10\rangle$
$^1C_3$	$\|(o^3\ \frac{1}{2}\ 21\ \ {}^221;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^221)\frac{1}{2}\ {}^10\rangle$
$^1C_4$	$\|(o^3\ \frac{1}{2}\ 21\ \ {}^421;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^421)\frac{1}{2}\ {}^10\rangle$

(b) Spin 1 $\quad ^3C$ $o^3\bar{o}^3$ states

$^3C_1$	$\|(o^3\ \frac{1}{2}\ 21\ {}^20\ ;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^20\ )\frac{1}{2}\ {}^30\rangle$
$^3C_2$	$\|(o^3\ \frac{1}{2}\ 21\ {}^23\ ;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^23\ )\frac{1}{2}\ {}^30\rangle$
$^3C_3$	$\|(o^3\ \frac{1}{2}\ 21\ {}^421;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^421)\frac{1}{2}\ {}^30\rangle$
$^3C_4$	$\|(o^3\ \frac{1}{2}\ 21\ {}^421;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^221)\frac{1}{2}\ {}^30\rangle$
$^3C_5$	$\|(o^3\ \frac{1}{2}\ 21\ {}^221;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^421)\frac{1}{2}\ {}^30\rangle$
$^3C_6$	$\|(o^3\ \frac{1}{2}\ 21\ {}^221;\ \bar{o}^3\ \frac{1}{2}\ 2^41\ \ {}^221)\frac{1}{2}\ {}^30\rangle$

All strangeness labels have been suppressed.

(5.3) THE MASS CALCULATION

In the MIT bag model a hadron is regarded as a volume of space in which the coloured quarks and gluons are confined by a constant energy density. A general solution for an arbitrarily shaped bag does not exist, however solutions do exist for two special cases: where the bag shape is approximated by a string (for high angular momentum hadrons), and where the bag is approximated to a static sphere (angular momentum zero hadrons). Only S-wave states are of interest here and so it is the spherical bag which is of interest.

In the spherical approximation the mass of the hadron is calculated as the sum of several terms: E_q, the quark kinetic and rest energy; E_V, the energy resulting from the confining pressure; E_O, an estimate of the effects of the zero-point fluctuations; and E_g, the energy associated with a one-gluon exchange between the quarks.

The quark energy term, E_q, is obtained by using standard techniques for particles in a cavity, giving

$$E_q = \frac{1}{R} \sum_i \omega(m_i R) \quad , \tag{5.3}$$

where R is the radius of the bag and m_i is the mass of the i^{th} quark. The term $\omega(m_i R)$ is the frequency of the lowest quark eigenmode in the bag and is given by

$$\omega(m_i R) = (x^2 + (m_i R)^2)^{\frac{1}{2}} \quad , \tag{5.4}$$

where $x = x(m_i R)$ is the smallest positive root of the eigenvalue equation

$$\tan x = \frac{x}{1-m_iR-(x+(m_iR)^2)^{\frac{1}{2}}} \quad . \tag{5.5}$$

The next term, E_V, produces the confinement of the quarks in the bag, and is given by

$$E_V = \frac{4}{3}\pi R^3 B \quad . \tag{5.6}$$

where B is the constant energy density to be fitted.

The third term in the sum, E_O, arises because the zero points of the modes cannot be subtracted out as they are in many applications. This is because the volume of the bag depends on the mass of the states. The zero-point energy is written as

$$E_O = -Z_O/R \quad , \tag{5.7}$$

with Z_O a parameter to be fitted.

The last term, E_g, provides the colour-hyperfine structure of the states. E_g can be further divided into a colour-electrostatic term, E_E, and a colour-magnetic term, E_M. DeGrand et al. (1975) indicate that E_E is always less than 5 MeV when the ordinary quarks are regarded as massless and the strange quark mass is less than 300 MeV. E_E will therefore be ignored in what follows. The colour-magnetic term is given by

$$E_M = -\frac{\alpha_c}{R} \sum_{i>j} \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\sigma}_j \; \underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j \; M(m_iR, m_jR), \tag{5.8}$$

where $\underset{\sim}{\lambda}_i$ is the colour and $\underset{\sim}{\sigma}_i$ is the spin of the ith quark. α_c is the quark-gluon coupling constant, the value of which is to be fitted. $M(m_i R,\ m_j R)$ is a radial integral, details of which are to be found in DeGrand et al. (1975).

To find the mass of a state the energy is minimised with respect to the bag radius, R. This is equivalent to balancing the 'external' pressure due to B with the quark and gluon field pressure.

The parameters fitted to known experimental masses are then the quark masses, m_o and m_s, the quark-gluon coupling constant, α_c, the bag pressure term, B, and the zero-point fluctuation factor, Z_o. DeGrand et al. (1975) obtained the following fit using the masses of the N, Δ, ω, and Ω:

$$
\begin{aligned}
m_o &= 0 \\
m_s &= 179 \text{ MeV} \\
\alpha_c &= 0.55 \\
B^{\frac{1}{4}} &= 146 \text{ MeV} \\
Z_o &= 1.84 \quad .
\end{aligned}
\tag{5.9}
$$

The most difficult task is the evaluation of the colour magnetic term E_m. Following Bickerstaff and Wybourne (1980), equation (5.8) may be rewritten as

$$E_M = \frac{\alpha_c}{R} \sum_{a \geqslant b} \Delta_g^{ab} M_{ab} \tag{5.10}$$

where

$$M_{ab} = M(m_a R, m_b R) \tag{5.11}$$

the operator is constructed out of the twenty-four single particle operators $\underset{\sim}{\sigma\lambda}$ which transform as $21^4\ {}^3 21$ under $SU_6^{CS} \downarrow SU_2^S \times SU_3^C$. These are coupled together to give a two particle operator X^{ab} which must transform as ${}^1 0$ under $SU_2^S \times SU_3^C$. The final form of the operator is calculated to be

$$\Delta_g^{ab} = [140\sqrt{6}(2 - \delta_{ab})X^{ab} + 8N\delta_{ab}], \tag{5.14}$$

where N is the number of quarks of flavour a. Using now standard results for coupled products of tensor operators (Butler 1975, 1980), the tables of reduced matrix elements in Wybourne and Bickerstaff (1980), and the few extra reduced matrix elements given in table (5.5), the matrix elements can be calculated relatively simply. In spite of the fact that the matrices had dimensions up to 14 with some elements having eight terms in the summation of (5.10), most matrix elements were calculated by hand, although three matrices were calculated using WIGNER, a computer program developed by Reid and Butler (1979, unpublished). The colour-spin matrix elements were then checked using Bickerstaff and Wybourne's checking method.

After the matrix elements are calculated it is a simple matter to evaluate all the $q^3\bar{q}^3$ S-wave states. This has been done using a computer program developed expressly for this purpose by Bickerstaff (1980). The results of these calculations are presented, according to total spin in table (5.4). States which are merely antiparticles of listed states have been omitted. The bag radii are given in terms of mR where m is the strange quark mass. The baryon-antibaryon channels to which each state couples have also been included for east of identification.

The question of the accuracy of the masses should be considered. Clearly the disregarding of the E_E term puts a lower limit on the uncertainty. However there is also another process which may affect the results. This is the effect of quark-antiquark virtual annihilation into gluons which is believed to be largely responsible for poor results in bag model calculations for the η-η' psuedoscalar mesons. In the $q\bar{q}$ configuration the effect is limited because quarks cannot annihilate into a single gluon but only into two (J=0 case) or three (J = 1 case) gluons (Jaffe 1977a). In multiquark mesons however, the presence of colour-octet, vector meson subsystems means that annihilation into one gluon is possible. While this effect is first order in α_c, as is gluon exchange, it is limited by a numerical factor, namely the fraction of the state which is of the appropriate type. This factor is small and hence, following Jaffe (1977), the effect has been ignored. Recent results by Bickerstaff (1982b), however, indicate that interesting physics is being lost, and care should be taken in the interpretation of the results listed in table (5.4).

(5.4) DISSOCIATION OF STATES

The $q^3\bar{q}^3$ configuration is special because it is the smallest multiquark configuration which has the choice of

TABLE 5.4(a)

$q^3\bar{q}^3$ states $J^P = 0^-$

matrix name	dimension	SU_6 content	Species	Isospin	S	$S\bar{S}$ pairs	(Radius(mR),Mass(MeV))	$B\bar{B}$ content
^1A	2	$1^3,1^3$	$O^3\bar{O}^3$	3,2,1,0	0	0	(1.78,1976),(1.93,2500)	$\Delta\bar{\Delta}$
			$S^3\bar{O}^3$	$\frac{3}{2}$	−3	0	(1.75,2528),(1.90,2926)	$\Omega\bar{\Delta}$
			$S^3\bar{S}^3$	0	0	3	(1.73,3049),(1.87,3359)	$\Omega\bar{\Omega}$
^1B	1	$1^3,2^4 1$	$O^3\bar{O}^3$	2,1	0	0	(1.86,2243)	-
			$S^3\bar{O}^3$	$\frac{1}{2}$	−3	0	(1.83,2688)	-
^1C	4	$21,2^4 1$	$O^3\bar{O}^3$	1,0	0	0	(1.59,1402),(1.77,1933),(1.83,2158),(1.89,2359)	$N\bar{N}$
^1D	8	$1^2,1,1^4,1^5$	$O^2S\bar{O}^2\bar{S}$	2,1,0	0	1	(1.57,1849),(1.75,2329),(1.76,2335),(1.82,2505),(1.84,2602),(1.84,2605), (1.88,2745),(1.91,2785)	$\Sigma^*\bar{\Sigma}^*,\Sigma\bar{\Sigma}$
			$S^2O\bar{O}^2\bar{S}$	$\frac{3}{2},\frac{1}{2}$	−1	1	(1.56,2067),(1.74,2489),(1.76,2515),(1.81,2647),(1.83,2711),(1.83,2749), (1.86,2817),(1.90,2926)	$\Xi^*\bar{\Sigma}^*,\Xi\bar{\Sigma}$
			$S^2O\bar{S}^2\bar{O}$	1,0	0	2	(1.56,2254),(1.73,2642),(1.75,2692),(1.80,2785),(1.82,2855),(1.82,2857), (1.84,2909),(1.89,3069)	$\Xi^*\bar{\Xi}^*,\Xi\bar{\Xi}$
^1E	8	$2,1,2^5,1^5$	$O^2S\bar{O}^2\bar{S}$	0	0	1	(1.24,1215),(1.57,1789),(1.67,2046),(1.69,2145),(1.70,2177),(1.75,2272), (1.81,2436),(1.87,2634)	$\Lambda\bar{\Lambda}$
^1F	7	$1^2,1,2^5,1^5$	$O^2S\bar{O}^2\bar{S}$	1	0	1	(1.57,1848),(1.69,2124),(1.75,2305),(1.82,2466),(1.83,2554),(1.87,2662), (1.87,2668)	$\Sigma\bar{\Lambda}$
			$S^2O\bar{O}^2\bar{S}$	$\frac{1}{2}$	−1	1	(1.56,2017),(1.69,2340),(1.74,2451),(1.81,2595),(1.83,2692),(1.86,2787), (1.86,2801)	$\Xi\bar{\Lambda}$
^1G	3	$1^2,1,1^3$	$O^2S\bar{O}^3$	$\frac{5}{2},\frac{3}{2},\frac{1}{2}$	−1	0	(1.77,2157),(1.85,2454),(1.92,2643)	$\Sigma^*\bar{\Delta}$
			$S^2O\bar{O}^3$	2,1	−2	0	(1.76,2341),(1.84,2554),(1.91,2784)	$\Xi^*\bar{\Delta}$
			$S^2O\bar{S}^3$	$\frac{1}{2}$	1	2	(1.74,2872),(1.81,3006),(1.88,3213)	$\Xi^*\bar{\Omega}$
			$O^2S\bar{S}^3$	1	2	1	(1.75,2698),(1.83,2893),(1.89,3069)	$\Sigma^*\bar{\Omega}$
^1H	5	$1^2,1,2^4 1$	$O^2S\bar{O}^3$	$\frac{3}{2},\frac{1}{2}$	−1	0	(1.58,1646),(1.76,2132),(1.83,2327),(1.84,2412),(1.88,2527)	$\Sigma\bar{N}$
			$S^2O\bar{O}^3$	1,0	−2	0	(1.57,1827),(1.75,2285),(1.82,2449),(1.84,2554),(1.87,2655)	$\Xi\bar{N}$
^1I	2	$2,1,1^3$	$O^2S\bar{O}^3$	$\frac{3}{2}$	−1	0	(1.85,2375),(1.88,2543)	-
			$O^2S\bar{S}^3$	0	2	1	(1.82,2825),(1.85,2923)	-
^1J	5	$2,1,2^4 1$	$O^2S\bar{O}^3$	$\frac{1}{2}$	−1	0	(1.58,1602),(1.70,1964),(1.76,2104),(1.82,2297),(1.88,2499)	$\Lambda\bar{N}$

TABLE 5.4(b) $q^3\bar{q}^3$ states $J^P = 1^-$

matrix name	dimension	SU_6 content	Species	Isospin	S	$\bar{s}s$ pairs	(Radius(mR), Mass(MeV))	$B\bar{B}$ content
^3A	2	$1^3,1^3$	$o^3\bar{o}^3$ $s^3\bar{o}^3$ $s^3\bar{s}^3$	3,2,1,0 $\frac{3}{2}$ 0	0 -3 0	0 0 3	(1.86,2247),(1.96,2619) (1.83,2734),(1.93,3016) (1.80,3211),(1.90,3428)	$\Delta\bar{\Delta}$ $\Omega\bar{\Delta}$ $\Omega\bar{\Omega}$
^3B	3	$1^3,2^41$	$o^3\bar{o}^3$ $s^3\bar{o}^3$	2,1 $\frac{1}{2}$	0 -3	0 0	(1.77,1944),(1.85,2234),(1.92,2482) (1.74,2453),(1.83,2678),(1.89,2876)	$\Delta\bar{N}$ $\Omega\bar{N}$
^3C	6	$21,2^41$	$o^3\bar{o}^3$	1,0	0	0	(1.68 , 1669),(1.73,1815),(1.78,1964),(1.81,2085),(1.84,2176),(1.93,2525)	$N\bar{N}$
^3D	14	$1^2,1,1^4,1^5$	$o^2s\bar{o}^2\bar{s}$	2,1,0	0	1	(1.67,2115),(1.71,2200),(1.75,2314),(1.75,2323),(1.76,2360),(1.80,2443), (1.83,2520),(1.84,2555),(1.84,2568),(1.84,2587),(1.90,2766),(1.90,2768), (1.91,2827),(1.94,2881)	$\Sigma^*\bar{\Sigma}^*,\Sigma^*\bar{\Sigma}$ $\Sigma\bar{\Sigma}^*,\Sigma\bar{\Sigma}$
			$s^2o\bar{o}^2\bar{s}$	$\frac{3}{2},\frac{1}{2}$	-1	1	(1.66,2274),(1.71,2379),(1.75,2483),(1.75,2493),(1.75,2516),(1.79,2602), (1.81,2676),(1.83,2710),(1.83,2724),(1.83,2747),(1.89,2900),(1.90,2921), (1.90,2945),(1.93,3015)	$\Xi^*\bar{\Sigma}^*,\Xi^*\bar{\Sigma},$ $\Xi\bar{\Sigma}^*,\Xi\bar{\Sigma}$
			$s^2o\bar{s}^2\bar{o}$	1,0	0	2	(1.65,2417),(1.70,2564),(1.74,2655),(1.74,2667),(1.74,2693),(1.78,2762), (1.80,2818),(1.82,2867),(1.82,2878),(1.82,2901),(1.88,3053),(1.88,3055), (1.89,3070),(1.92,3150)	$\Xi^*\bar{\Xi}^*,\Xi^*\bar{\Xi},$ $\Xi\bar{\Xi}^*,\Xi\bar{\Xi}$
^3E	14	$2,1,2^51^6$	$o^2s\bar{o}^2\bar{s}$	0	0	1	(1.56,1734),(1.57,1805),(1.57,1864),(1.66,2017),(1.72,2178),(1.73,2232), (1.74,2234),(1.75,2324),(1.78,2401),(1.80,2417),(1.82,2488),(1.84,2547), (1.84,2548),(1.91,2780)	$\Lambda\bar{\Lambda}$
^3F	13	$1^2,1,2^5,1^5$	$o^2s\bar{o}^2\bar{s}$	1	0	1	(1.58,1875),(1.67,2073),(1.71,2190),(1.75,2272),(1.76,2305),(1.76,2328), (1.76,2366),(1.79,2435),(1.82,2502),(1.84,2559),(1.84,2582),(1.90,2765), (1.91,2805)	$\Sigma^*\bar{\Lambda},\Sigma\bar{\Lambda}$
			$s^2o\bar{o}^2\bar{s}$	$\frac{1}{2}$	-1	1	(1.56,1992),(1.66,2218),(1.71,2368),(1.73,2390),(1.74,2468),(1.75,2498), (1.75,2509),(1.79,2589),(1.81,2649),(1.83,2703),(1.83,2709),(1.89,2894), (1.90,2927)	$\Xi^*\bar{\Lambda},\Xi\bar{\Lambda}$
^3G	5	$1^2,1,1^3$	$o^2s\bar{o}^3$ $s^2o\bar{o}^3$ $s^2o\bar{s}^3$ $o^2s\bar{s}^3$	$\frac{5}{3},\frac{3}{2},\frac{1}{2}$ 2,1 $\frac{1}{2}$ 1	-1 -2 1 2	0 0 2 1	(1.76,2146),(1.85,2398),(1.85,2427),(1.91,2637),(1.95,2749) (1.75,2338),(1.84,2561),(1.84,2592),(1.90,2790),(1.94,2882) (1.73,2834),(1.81,3020),(1.81,3054),(1.87,3191),(1.91,3288) (1.73,2654),(1.82,2860),(1.82,2895),(1.88,3038),(1.92,3151)	$\Sigma^*\bar{\Delta},\Sigma\bar{\Delta}$ $\Xi^*\bar{\Delta},\Xi\bar{\Delta}$ $\Xi^*\bar{\Omega},\Xi\bar{\Omega}$ $\Sigma^*\bar{\Omega},\Sigma\bar{\Omega}$
^3H	9	$1^2,1,2^41$	$o^2s\bar{o}^3$	$\frac{3}{2},\frac{1}{2}$	-1	0	(1.67,1898),(1.72,2008),(1.76,2118),(1.77,2157),(1.80,2265),(1.83,2346), (1.84,2391),(1.91,2610),(1.92,2674)	$\Sigma^*\bar{N},\Sigma\bar{N}$
			$s^2o\bar{o}^3$	1,0	-2	0	(1.66,2047),(1.72,2191),(1.76,2285),(1.76,2321),(1.80,2421),(1.82,2500) (1.83,2536),(1.90,2741),(1.91,2792)	$\Xi^*\bar{N},\Xi\bar{N}$

TABLE 5.4(b) cont. $q^3\bar{q}^3$ states $J^P = 1^-$ Continued

matrix name	dimension	SU_6 Content	Species	Isospin	S	$s\bar{s}$ pairs	(Radius(mR), Mass(MeV))	$B\bar{B}$ content
^3I	4	$2,1,1^3$	$O^2 s\bar{O}^3$	$\frac{3}{2}$	-1	0	(1.76,2138),(1.77,2173),(1.85,2406),(1.91,2635)	$\Lambda\bar{\Delta}$
			$O^2 s\bar{s}^3$	0	2	1	(1.74,2636),(1.74,2638),(1.82,2847),(1.88,3031)	$\Lambda\bar{\Omega}$
^3J	9	$2,1,2^4 1$	$O^2 s\bar{O}^3$	$\frac{1}{2}$	-1	0	(1.58,1644),(1.67,1845),(1.72,2003),(1.75,2090),(1.77,2148),(1.80,2251) (1.83,2336),(1.85,2413),(1.92,2651)	$\Lambda\bar{N}$

TABLE 5.4(c)

$q^3\bar{q}^3$ states $J^P = 2^-$

matrix name	dimension	SU_6 Content	Species	Isospin	S	$s\bar{s}$ pairs	(Radius(mR), Mass (MeV))	$B\bar{B}$ content
^5A	1	$1^3,1^3$	$o^3\bar{o}^3$	3,2,1,0	0	0	(1.89,2359)	$\Delta\bar{\Delta}$
			$s^3\bar{o}^3$	$\frac{3}{2}$	-3	0	(1.86,2819)	$\Omega\bar{\Delta}$
			$s^3\bar{s}^3$	0	0	3	(1.83,3277)	$\Omega\bar{\Omega}$
^5B	2	$1^3,2^41$	$o^3\bar{o}^3$	2,1	0	0	(1.81,2065),(1.89,2359)	$\Delta\bar{N}$
			$s^3\bar{o}^3$	$\frac{1}{2}$	-3	0	(1.78,2541),(1.86,2786)	$\Omega\bar{N}$
^5C	3	$21,2^41$	$o^3\bar{o}^3$	1,0	0	0	(1.77,1929),(1.84,2198),(1.89,2359)	-
^5D	8	$1^2,1,1^4,1^5$	$o^2s\bar{o}^2\bar{s}$	2,1,0	0	1	(1.76,2349),(1.79,2439),(1.79,2449),(1.83,2511),(1.87,2643),(1.87,2666), (1.87,2687),(1.87,2689)	$\Sigma^*\bar{\Sigma}^*$, $\Sigma^*\bar{\Sigma}$, $\Sigma\bar{\Sigma}^*$
			$s^2o\bar{o}^2\bar{s}$	$\frac{3}{2},\frac{1}{2}$	-1	1	(1.74,2473),(1.78,2591),(1.78,2601),(1.82,2694),(1.86,2787),(1.86,2807), (1.86,2825),(1.86,2833)	$\Xi^*\bar{\Sigma}^*$, $\Xi^*\bar{\Sigma}$, $\Xi\bar{\Sigma}^*$
			$s^2o\bar{s}^2\bar{o}$	1,0	0	2	(1.73,2586),(1.77,2738),(1.77,2744),(1.81,2872),(1.85,2942),(1.85,2947) (1.85,2971),(1.85,2983)	$\Xi^*\bar{\Xi}^*$, $\Xi^*\bar{\Xi}$, $\Xi\bar{\Xi}^*$
^5E	8	$2,1,2^5,1^5$	$o^2s\bar{o}^2\bar{s}$	0	0	1	(1.74,2206),(1.74,2234),(1.75,2331),(1.75,2362),(1.83,2547),(1.87,2640), (1.87,2662),(1.87,2665)	-
^5F	7	$1^2,1,2^5,1^5$	$o^2s\bar{o}^2\bar{s}$	1	0	1	(1.75,2274),(1.75,2302),(1.79,2394),(1.83,2528),(1.87,2645),(1.87,2675), (1.87,2675	$\Sigma^*\bar{\Lambda}$
			$s^2o\bar{o}^2\bar{s}$	$\frac{1}{2}$	-1	1	(1.74,2403),(1.75,2499),(1.78,2550),(1.82,2709),(1.86,2787),(1.86,2801), (1.86,2811)	$\Xi^*\bar{\Lambda}$
^5G	3	$1^2,1,1^3$	$o^2s\bar{o}^3$	$\frac{5}{2}$ $\frac{3}{2}$ $\frac{1}{2}$	-1	0	(1.80,2284),(1.88,2501),(1.88,2543)	$\Sigma^*\bar{\Delta}$, $\Sigma\bar{\Delta}$
			$s^2o\bar{o}^3$	2,1	-2	0	(1.79,2422),(1.87,2651),(1.87,2688)	$\Xi^*\bar{\Delta}$ $\Xi\bar{\Delta}$
			$s^2o\bar{s}^3$	$\frac{1}{2}$	1	2	(1.76,2898),(1.84,3098),(1.84,3129)	$\Xi^*\bar{\Omega}$, $\Xi\bar{\Omega}$
			$o^2s\bar{s}^3$	1	2	1	(1.77,2755),(1.85,2951),(1.85,2979)	$\Sigma^*\bar{\Omega}$, $\Sigma\bar{\Omega}$
^5H	5	$1^2,1,2^41$	$o^2s\bar{o}^3$	$\frac{3}{2},\frac{1}{2}$	-1	0	(1.76,2140),(1.79,2231),(1.84,2352),(1.88,2495),(1.88,2522)	$\Sigma^*\bar{N}$
			$s^2o\bar{o}^3$	1,0	-2	0	(1.75,2256),(1.79,2386),(1.83,2538),(1.87,2637),(1.87,2660)	$\Xi^*\bar{N}$
^5I	2	$2,1,1^3$	$o^2s\bar{o}^3$	$\frac{3}{2}$	-1	0	(1.80,2229),(1.88,2510)	$\Lambda\bar{\Delta}$
			$o^2s\bar{s}^3$	0	2	1	(1.77,2706),(1.85,2938)	$\Lambda\bar{\Omega}$
^5J	5	$2,1,2^41$	$o^2s\bar{o}^3$	$\frac{1}{2}$	-1	0	(1.76,2071),(1.76,2144),(1.84,2375),(1.88,2501),(1.88,2526)	-

TABLE 5.4(d)

$q^3\bar{q}^3$ states $J^P = 3^-$

matrix name	dimension	SU_6 Content	Species	Isospin	S	$\bar{s}s$ pairs	(Radius (mR), Mass (MeV))	$B\bar{B}$ content
^7A	1	$1^3,1^3$	$O^3\bar{O}^3$	3,2,1,0	0	0	(1.89,2359)	$\Delta\bar{\Delta}$
			$S^3\bar{O}^3$	$\frac{3}{2}$	-3	0	(1.86,2819)	$\Omega\bar{\Delta}$
			$S^3\bar{S}^3$	0	0	3	(1.83,3277)	$\Omega\bar{\Omega}$
^7C	1	$21,2^41$	$O^3\bar{O}^3$	1,0	0	0	(1.89,2359)	-
^7D	2	$1^2,1,1^4,1^5$	$O^2S\bar{O}^2\bar{S}$	2,1,0	0	1	(1.87,2660),(1.87,2666)	$\Sigma^*\bar{\Sigma}^*$
			$S^2O\bar{O}^2\bar{S}$	$\frac{3}{2},\frac{1}{2}$	-1	1	(1.86,2812),(1.86,2816)	$\Xi^*\bar{\Sigma}^*$
			$S^2O\bar{S}^2\bar{O}$	1,0	0	2	(1.85,2965),(1.85,2972)	$\Xi^*\bar{\Xi}^*$
^7E	2	$2,1,2^51^5$	$O^2S\bar{O}^2\bar{S}$	0	0	1	(1.87,2661),(1.87,2664)	-
^7F	1	$1^2,1,2^5,1^5$	$O^2S\bar{O}^2\bar{S}$	1	0	1	(1.87,2660)	-
			$S^2O\bar{O}^2\bar{S}$	$\frac{1}{2}$	-1	1	(1.86,2816)	-
^7G	1	$1^2,1,1^3$	$O^2S\bar{O}^3$	$\frac{5}{2},\frac{3}{2},\frac{1}{2}$	-1	0	(1.88,2510)	$\Sigma^*\bar{\Delta}$
			$S^2O\bar{O}^3$	2,1	-2	0	(1.87,2663)	$\Xi^*\bar{\Delta}$
			$S^2O\bar{S}^3$	$\frac{1}{2}$	1	2	(1.84,3122)	$\Xi^*\bar{\Omega}$
			$O^2S\bar{S}^3$	1	2	1	(1.85,2969)	$\Sigma^*\bar{\Omega}$
^7H	1	$1^2,1,2^41$	$O^2S\bar{O}^3$	$\frac{3}{2},\frac{1}{2}$	-1	0	(1.88,2510)	-
			$S^2O\bar{O}^3$	1,0	-2	0	(1.87,2663)	-
^7J	1	$2,1,2^41$	$O^2S\bar{O}^3$	$\frac{1}{2}$	-1	0	(1.88,2510)	-

TABLE (5.5): Necessary $SU_6 \downarrow SU_2 \times SU_3$ reduced matrix elements additional to Bickerstaff and Wybourne (1980)

$$<21\ {}^{2}3 \| T \| 21\ {}^{2}0> = 0$$

$$<21\ {}^{2}3 \| T \| 21\ {}^{2}21>^{-} = -\frac{4}{7}\sqrt{7}$$

$$<21\ {}^{2}3 \| T \| 21\ {}^{4}21>^{+} = -\frac{4}{7}\sqrt{7}$$

$$<21\ {}^{2}3 \| T \| 21\ {}^{2}3> = \frac{2}{7}\sqrt{14}$$

Where $T = 21^{4}\ {}^{3}21$ and a + or - superscript indicates whether or not a matrix changes sign under permutation.

two dissociation channels. Many states may dissociate into either baryon-antibaryon, $(q^3)(\bar{q}^3)$, or three meson, $(q\bar{q})^3$, channels. Since the basis states found in section (5.3) are essentially $(q^3)(\bar{q}^3)$ coupled states, the fraction of each bag-model state which is baryon-antibaryon may be trivially obtained, given the eigenvectors of the bag-model Hamiltonian. To find the $(q\bar{q})^3$ content of the states a basis transformation must be made on the states of section (5.3). The fraction of the bag-model state which is $(q\bar{q})^3$ can then be found, given the transformation factors and the energy eigenvectors.

The general method for calculating such a type of basis transformation is found in Bickerstaff and Wybourne (1981). In this approach the transformation is done in several steps. For $q^3\bar{q}^3$ states the first step is to decouple a quark from the q^3 part and an antiquark from the $\bar{q}^3$ part to achieve $(q^2q)(\bar{q}^2\bar{q})$ label states. Then a transformation which interchanges the order of coupling is performed to produce $(q^2\bar{q}^2)(q\bar{q})$ labelled states. By a similar process, the $q^2\bar{q}^2$ part can be dissociated into $(q\bar{q})^2$ labelled states to obtain $(q\bar{q})^3$ (actually $(q\bar{q})^2(q\bar{q})$) labelled states.

Dissociation calculations have been performed on the (^1C and ^3C) nucleon-antinucleon associated states. These involve no strangeness related labels (table 5.3) and so SU_{12} is the largest group required. The first step of the transformation, from q^3 to q^2q, using $SU_{12} \downarrow SU_2^I \times [SU_6^{CS} \downarrow SU_2^S \times SU_3^C]$ labelled states, can be written as

$$|(1^2,1)1^3(I\lambda^{CS}S\mu^C)>$$

$$= \sum|[1^2(I'\lambda'S'\mu');1(I''\lambda''S''\mu'')]IS\mu^C>$$

$$\times <[1^2(I'\lambda');1(I''\lambda'')]I\lambda^{CS}|(1^2,1)1^3(I\lambda^{CS})>$$

$$\times <(\lambda'S'\mu';\lambda''S''\mu'')S\mu^C|(\lambda',\lambda'')\lambda^{CS}S\mu^C>, \qquad (5.15)$$

where the summation is over all primed and double-primed symbols. The second step, from $(q^2q)(\bar{q}^2\bar{q})$ to $(q^2\bar{q}^2)(q\bar{q})$ is given by

$$(I_{q^2}S_{q^2}\mu_{q^2},I_qS_q\mu_q)I_{q^3}S_{q^3}\mu_{q^3},$$

$$(I_{\bar{q}^2}S_{\bar{q}^2}\mu_{\bar{q}^2},I_{\bar{q}}S_{\bar{q}}\mu_{\bar{q}})I_{\bar{q}^3}S_{\bar{q}^3}\mu_{\bar{q}^3};ISO^C>$$

$$= \sum|(I_{q^2}S_{q^2}\mu_{q^2},I_{\bar{q}^2}S_{\bar{q}^2}\mu_{\bar{q}^2})I'_{q^2\bar{q}^2}S'_{q^2\bar{q}^2}\mu'_{q^2\bar{q}^2},$$

$$(I_qS_q\mu_q,I_{\bar{q}}S_{\bar{q}}\mu_{\bar{q}})I''_{q\bar{q}}S''_{q\bar{q}}\mu''_{q\bar{q}};ISO^C>$$

$$\times <((I_{q^2},I_{\bar{q}^2})I'_{q^2\bar{q}^2},(I_q,I_{\bar{q}})I''_{q\bar{q}})I|((I_{q^2},I_q)I_{q^3},(I_{\bar{q}^2},I_{\bar{q}})I_{\bar{q}^3})I>$$

$$\times <((S_{q^2},S_{\bar{q}^2})S'_{q^2\bar{q}^2},(S_qS_{\bar{q}})S''_{q\bar{q}})S|((S_{q^2},S_q)S_{q^3},(S_{\bar{q}^2},S_{\bar{q}})S_{\bar{q}^3})S>$$

$$\times <((\mu_{q^2},\mu_{\bar{q}^2})\mu'_{q^2\bar{q}^2},(\mu_q,\mu_{\bar{q}})\mu''_{q\bar{q}})O|((\mu_{q^2},\mu_q)\mu_{q^3},(\mu_{\bar{q}^2},\mu_{\bar{q}})\mu_{\bar{q}^3})O>. \qquad (5.16)$$

where parentage labels have been suppressed. Similar formulae apply for the $q^2\bar{q}^2$ to $(q\bar{q})^2$ dissociation.

Evaluation of these formulae was done using the more symmetric 3jm factors and 6j symbols (Butler 1975, 1981). Tables of the necessary $SU_{12} \downarrow SU_2 \times SU_6$ 3jm factors can be found in Bickerstaff and Wybourne (1981). Also required were SU_3 6j symbols (Bickerstaff, Butler, Butts, Haase and Reid 1982) and SU_2 6j symbols (Rotenberg et al. 1959).

The results of the dissociation calculations the ^1C and ^3C states are listed in tables (5.6) and (5.7). The fraction of the states which has coloured components in the dissociation is labelled 'colour' and can be ignored, except possibly when some sort of comparison is being made between fractions of $B\bar{B}$ and 3m dissociations. There is no guarantee, however, that table (5.6) fractions can be compared to table (5.7) fractions.

The $N\bar{N}$ related states and the eigenvectors used in these calculations are explicitly presented in table (5.8).

(5.5) COMMENTS AND CONCLUSIONS

Perhaps the most obvious feature of the results of this work on $q^3\bar{q}^3$ states is the large number of states produced from just three quarks and three antiquarks. The colour-hyperfine interaction is largely responsible for this as it can split a state into as many as 14 widely separated masses (^3D states). Another feature is the large number of degenerate isospin multiplets, although no doubt many of these would disappear if the mechanism that mixes the η and η' in the $q\bar{q}$ sector was included.

TABLE 5.6: $B\bar{B}$ dissociations

(a) ^1C States

STATE	$N\bar{N}$ CONTENT	COLOUR CONTENT
1402	.147	.853
1933	.799	.201
2158	.001	.999
2359	.052	.947

(b) ^3C States

STATE	$N\bar{N}$ CONTENT	COLOUR CONTENT
1669	.430	.570
1815	.002	.998
1964	.372	.628
2085	.000	1.000
2176	.194	.805
2525	.001	.999

TABLE 5.7: 3 meson dissociations

(a) ^1C states (I = 0)

CONTENT	STATE 1402	1933	2158	2359
$\rho\rho\rho$	.000	.000	.000	.176
$\rho\rho\eta_o$	.005	.058	.000	.044
$\rho\pi\omega$	.010	.022	.110	.036
$\pi\pi\eta_o$	.075	.005	.000	.001
$\omega\omega\eta_o$	.002	.000	.078	.001
$\eta_o\eta_o\eta_o$	.075	.005	.000	.001
Colour	.834	.910	.812	.741

(b) ^1C states (I = 1)

CONTENT	STATE 1402	1933	2158	2359
$\pi\pi\pi$	.124	.009	.000	.002
$\rho\rho\pi$	.012	.053	.151	.055
$\rho\omega\eta_o$	.003	.007	.037	.012
$\pi\eta_o\eta_o$	.025	.002	.000	.000
$\omega\omega\pi$	.002	.019	.000	.015
$\rho\rho\omega$	.000	.000	.000	.176
Colour	.834	.910	.812	.741

(c) ^3C states (I = 0)

CONTENT	STATE 1669	1815	1964	2085	2176	2525
$\rho\rho\omega$	.001	.005	.003	.000	.060	.192
$\omega\omega\omega$	.001	.003	.003	.000	.017	.061
$\rho\pi\eta_o$	.012	.088	.021	.000	.003	.001
$\omega\pi\pi$	.088	.000	.026	.000	.001	.002
$\omega\eta_o\eta_o$	.007	.064	.001	.000	.005	.000
$\rho\pi\omega$	.000	.000	.000	.148	.000	.000
$\rho\rho\pi$	.000	.004	.104	.000	.051	.017
Colour	.891	.835	.842	.852	.854	.727

(d) ^3C states (I = 1)

CONTENT	STATE 1669	1815	1964	2085	2176	2525
$\rho\rho\rho$	.002	.007	.005	.000	.063	.189
$\rho\pi\pi$	.074	.123	.032	.000	.008	.001
$\rho\omega\omega$	.000	.002	.001	.000	.022	.064
$\rho\rho\eta_o$	.000	.001	.035	.000	.017	.006
$\rho\omega\eta_o$	.000	.000	.000	.049	.000	.000
$\rho\eta_o\eta_o$	.030	.000	.009	.000	.001	.001
$\pi\omega\eta_o$	.004	.030	.001	.000	.001	.000
$\rho\rho\pi$	.000	.000	.000	.099	.000	.000
$\rho\pi\omega$	.000	.003	.070	.000	.034	.011
Colour	.891	.835	.842	.852	.854	.727

TABLE 5.8: Nucleon-antinucleon related states and eigenvectors

(a) J = 0 States

QUANTUM NUMBERS ($J^{PC}(I^G)$)	ENERGY EIGENSTATES (MeV)	BASIS STATES ($SU_2 \times SU_3$) $^{4}21,^{4}21$	$^{2}21,^{2}21$	$^{2}0,^{2}0$	$^{2}3,^{2}3$
$0^{-+}(0^+),0^{-+}(1^-)$	1402	-.604	-.576	-.384	-.394
$0^{-+}(0^+),0^{-+}(1^-)$	1933	-.232	-.099	.894	-.371
$0^{-+}(0^+),0^{-+}(1^-)$	2158	.763	-.487	-.033	-.425
$0^{-+}(0^+),0^{-+}(1^-)$	2359	.000	.649	-.229	-.725

(b) J = 1 States

QUANTUM NUMBERS ($J^{PC}(I^G)$)	ENERGY EIGENSTATES (MeV)	BASIS STATES ($SU_2 \times SU_3$) $^{4}21,^{4}21$	$^{4}21,^{2}21$	$^{2}21,^{4}21$	$^{2}21,^{2}21$	$^{2}0,^{2}0$	$^{2}3,$
$1^{--}(0^-),1^{--}(1^+)$	1699	-.324	.461	-.461	-.013	-.656	.20
$1^{--}(0^-),1^{--}(1^+)$	1815	.764	.184	-.184	.521	-.044	.27
$1^{--}(0^-),1^{--}(1^+)$	1964	.000	.323	-.323	-.431	.610	.48
$1^{-+}(0^+),1^{-+}(1^-)$	2085*	.000	.707	.707	.000	.000	.00
$1^{--}(0^-),1^{--}(1^+)$	2176	.424	-.239	.239	-.649	-.441	.29
$1^{--}(0^-),1^{--}(1^+)$	2525	-.363	-.303	.303	.349	-.025	.75

*$^{3}C(2085)$ has no $\overline{NN}$ Content (See table 5(b))

Looking at just the nucleon-antinucleon related states, no multiquark states are seen to exist which cannot dissociate into three mesons of less total mass than the multiquark state. There are, however, many states bound to their $N\bar{N}$ channels. Naively one could see a possible decay mechanism in this for $N + \bar{N} \rightarrow 3m$ collisions. However the presence of the less massive 3 meson states invalidates the static spherical approximation. This means that there is no solid theoretical basis for suggesting that the $q^3\bar{q}^3$ states will appear as a state or resonance in any physical situation.

It is interesting to note however, that the predicted $N\bar{N}$ $q^3\bar{q}^3$ states are exactly in the regions of experimental uncertainty, that is the S(1936), the T(2200) and U(2400) meson regions. This gives some indication that further study may be profitable in future. However, at the present time, given the state of experimental uncertainty and theoretical ambiguity, there seems little point in pursuing explicit calculations on this configuration.

APPENDIX I

S-FUNCTION IDENTITIES

The identities (3.27) and (3.28) in section (3.3) require some justification, however they may be proved using simple arguments. Firstly consider (3.27a). Using (3.2) it may be seen that

$$Z \times \{\bar{\mu};\lambda\} = \{\bar{0};Z\} \times \{\bar{\mu};\lambda\}$$

$$= \sum_{\zeta} \{\overline{\mu/\zeta};\lambda\cdot(Z/\zeta)\} \ . \qquad \text{(I.1)}$$

If Z is a series made up of only one-row or one-column partitions, then (ζ) must be such a partition. Moreover the resulting S-functions from (Z/ζ) must also be of such a type. The series involved are the L,M,P and Q series. Hence for these series, (I.1) may be written as

$$Z.\{\bar{\mu};\lambda\} = \{\overline{\mu/Z};\lambda\cdot Z\} \ , \qquad \text{(I.2)}$$

which is just (3.27a). Since V = LQ and W = MP, (I.2) must also be true for V and W as well. The proof for (3.27b) is almost identical.

The identity (3.28) is not a series identity as such, since it may be stated as

$$\sum_{\zeta} \{\overline{\zeta/\nu};(\lambda/\zeta)\cdot\rho\} = \sum_{\eta} \{\bar{\eta};(\lambda/\eta\cdot\nu)\cdot\rho\} \ . \qquad \text{(I.3)}$$

While this may look complicated, it is easy to prove by using (2.1) and (2.2) in the following way

$$\sum_{\zeta} \{\overline{\zeta/\nu};(\lambda/\zeta)\cdot\rho\} = \sum_{\rho,\eta} m^{\zeta}_{\eta\nu}\{\bar{\eta};(\lambda/\zeta)\cdot\rho\} \quad \text{by} \quad (2.2)$$

$$= \sum_{\eta} \{\bar{\eta};(\lambda/\eta\cdot\nu)\cdot\rho\} \quad \text{by} \quad (2.1).$$

Another important set of identities involves the division of a product of S-functions by a series of S-functions (King et al. 1981). The rule for division of a product by a single S-function is given by (2.3) as

$$(\lambda.\mu)/\nu = \sum_{\alpha} (\lambda/\alpha).(\mu/(\nu/\alpha)) \tag{I.4}$$

Hence, for a general series Z, a formula may be written as

$$(\lambda\cdot\mu)/Z = \sum_{\alpha} (\lambda/\alpha)\cdot(\mu/(Z/\alpha)) \tag{I.5}$$

Using the same arguments that produced (I.2) from (I.1) this formula may be rewritten for the P,Q,L,M,V and W series as

$$(\lambda\cdot\mu)/Z = (\lambda/Z).(\mu/Z). \tag{I.6}$$

The production of similar formulae for the other series from first principles is a much more difficult task. However some may be derived from the Kronecker product formulae for O_n and Sp_{2k}. Rules (3.3) and (3.4) can only be equivalent if

$$(\lambda\cdot\mu)/Z = \sum_{\zeta} (\lambda/\zeta.Z).(\mu/\zeta\cdot Z) \tag{I.7}$$

for Z = B. The equivalence of (3.6) and (3.7) indicates that (I.7) is also true for Z = D. From (2.8) it can be seen that F = BM = DQ and that H = BP = DL and so (I.7) is true for Z = F and H as well.

The rule for the series A,C,E and G may be derived from (I.7) and is

$$(\lambda\cdot\mu)/Z = \sum_{\zeta} (-1)^{z}(\lambda/\zeta\cdot Z)\cdot(\mu/\tilde{\zeta}\cdot Z) \tag{I.8}$$

To prove this for these series it is necessary only to prove the case where Z = A since AL = E, AQ = G and ALQ = C. Division by the series product AB is a null operation but use of (I.7) leads to

$$\{\lambda\}\cdot\{\mu\} = \{\lambda\cdot\mu\}/AB = \sum_{\zeta} ((\lambda/\zeta\cdot B)\cdot(\mu/\zeta\cdot B))/A \ . \tag{I.9}$$

(I.8) is proved if

$$\{\lambda\cdot\mu\} = \sum_{\zeta,\eta} (-1)^{e}(\lambda/\zeta\cdot\eta)\cdot(\mu/\zeta\cdot\tilde{\eta}) \ , \tag{I.10}$$

where e is the weight of (η). This may be shown using the relation (King 1975b)

$$\sum_{\sigma} (-1)^{s}\{\pi/\sigma\}\cdot\{\tilde{\sigma}\} = \delta_{\pi o}\{0\} \ , \tag{I.11}$$

which may be expanded out using (2.1) and (2.2) to give

$$\sum_{\sigma,\tau,\nu} (-1)^{s} \, m^{\pi}_{\tau\sigma} \, m^{\nu}_{\tau\tilde{\sigma}}\{\nu\} = \delta_{\pi o}\{0\} \ . \tag{I.12}$$

(I.10) may be expanded similarly as

$$\{\lambda\cdot\mu\} = \sum_{\zeta,\eta,\alpha,\beta} (-1)^{e} \, m^{\alpha}_{\zeta\eta} \, m^{\beta}_{\zeta\tilde{\eta}} \, \{\sigma/\alpha\}\cdot\{\tau/\beta\} \ . \tag{I.13}$$

Substituting (I.12) in (I.13) gives

$$\{\lambda\cdot\mu\} = \sum_{\alpha} \delta_{\alpha o} \, \{\sigma/\alpha\}\cdot\{\tau\} \quad , \tag{I.14}$$

which is clearly true.

APPENDIX II: Branching rules for $SO_{10} \downarrow SU_5 \times U_1$

These rules are evaluated using the formulae of section (3.3). The U_1 factors have been calculated using the same methods as King (1981b).

$[0]$

$+\{0\}\{0\}$

$[S;0]+$

$+\{0\}\{5\}+\{1^3\}\{1\}+\{1\}\{-3\}$

$[S;0]-$

$+\{1^4\}\{3\}+\{1^2\}\{-1\}+\{0\}\{-5\}$

$[1]$

$+\{1\}\{2\}+\{1^4\}\{-2\}$

$[1^2]$

$+\{1^2\}\{4\}+\{2\,1^3\}\{0\}+\{0\}\{0\}+\{1^3\}\{-4\}$

$[1^3]$

$+\{1^3\}\{6\}+\{2^2\,1^2\}\{2\}+\{1\}\{2\}+\{2\,1^2\}\{-2\}$
$+\{1^4\}\{-2\}+\{1^2\}\{-6\}$

$[1^4]$

$+\{1^4\}\{8\}+\{2^3\,1\}\{4\}+\{1^2\}\{4\}+\{2^2\,1\}\{0\}$
$+\{2\,1^3\}\{0\}+\{0\}\{0\}+\{2\,1\}\{-4\}+\{1^3\}\{-4\}$
$+\{1\}\{-8\}$

$[1^5]+$

$+\{0\}\{20\}+\{1^3\}\{16\}+\{2^3\}\{12\}+\{1\}\{12\}$
$+\{2\,1^2\}\{8\}+\{2\}\{4\}$

$[1^5]-$

$+\{2^4\}\{16\}+\{2^2\,1^2\}\{12\}+\{2^2\}\{8\}+\{1^4\}\{8\}+\{1^2\}\{4\}$

$+\{0\}\{0\}$

$[S;1]+$

$+\{1\}\{7\}+\{2\,1^2\}\{3\}+\{1^4\}\{3\}+\{2^3\,1\}\{-1\}$
$+\{2\}\{-1\}+\{1^2\}\{-1\}+\{2\,1^3\}\{-5\}$

$[S;1]-$

$+\{2\,1^3\}\{5\}+\{2^4\}\{1\}+\{2\,1\}\{1\}+\{1^3\}\{1\}$
$+\{2^2\,1^2\}\{-3\}+\{1\}\{-3\}+\{1^4\}\{-7\}$

$[S;1^2]+$

$+\{1^2\}\{9\}+\{2^2\,1\}\{5\}+\{2\,1^3\}\{5\}+\{0\}\{5\}$
$+\{3\,2^2\,1\}\{1\}+\{2\,1\}\{1\}+2\{1^3\}\{1\}+\{3\,1^3\}\{-3\}$
$+\{2^3\}\{-3\}+\{2^2\,1^2\}\{-3\}+\{1\}\{-3\}+\{2\,1^2\}\{-7\}$

$[S;1^2]-$

$+\{2^2\,1^2\}\{7\}+\{3\,2^3\}\{3\}+\{2^2\}\{3\}+\{2\,1^2\}\{3\}$
$+\{1^4\}\{3\}+\{3\,2\,1^2\}\{-1\}+\{2^3\,1\}\{-1\}+2\{1^2\}\{-1\}$
$+\{2^2\,1\}\{-5\}+\{2\,1^3\}\{-5\}+\{0\}\{-5\}+\{1^3\}\{-9\}$

$[S;1^3]+$

$+\{1^3\}\{11\}+\{2^3\}\{7\}+\{2^2\,1^2\}\{7\}+\{1\}\{7\}$
$+\{3^2\,2\,1\}\{3\}+2\{2\,1^2\}\{3\}+\{1^4\}\{3\}+\{3\,2^2\}\{-1\}$
$+\{3\,2\,1^2\}\{-1\}+\{2^3\,1\}\{-1\}+\{2\}\{-1\}+\{1^2\}\{-1\}$
$+\{3\,1^2\}\{-5\}+\{2^2\,1\}\{-5\}+\{2\,1^3\}\{-5\}+\{2\,1\}\{-9\}$

$[S;1^3]-$

$+\{2^3\,1\}\{9\}+\{3^2\,2^2\}\{5\}+\{2^2\,1\}\{5\}+\{2\,1^3\}\{5\}$
$+\{3^2\,1^2\}\{1\}+\{3\,2^2\,1\}\{1\}+\{2^4\}\{1\}+\{2\,1\}\{1\}$

$+ \{1^3\}\{1\} + \{3\,2\,1\}\{-3\} + 2\{2^2\,1^2\}\{-3\} + \{1\}\{-3\}$

$+ \{2^2\}\{-7\} + \{2\,1^2\}\{-7\} + \{1^4\}\{-7\} + \{1^2\}\{-11\}$

$[S ; 1^4]+$

$+ \{1^4\}\{13\} + \{2^3\,1\}\{9\} + \{1^2\}\{9\} + \{3^3\,1\}\{5\}$

$+ \{2^2\,1\}\{5\} + \{2\,1^3\}\{5\} + \{0\}\{5\} + \{3\,2^2\}\{1\}$

$+ \{3\,2^2\,1\}\{1\} + \{2\,1\}\{1\} + \{1^3\}\{1\} + \{3\,2\,1\}\{-3\}$

$+ \{3\,1^3\}\{-3\} + \{2^3\}\{-3\} + \{1\}\{-3\} + \{3\,1\}\{-7\}$

$+ \{2\,1^2\}\{-7\} + \{2\}\{-11\}$

$[S ; 1^4]-$

$+ \{2^4\}\{11\} + \{3^3\,2\}\{7\} + \{2^2\,1^2\}\{7\} + \{3\,2^2\,1\}\{3\}$

$+ \{3\,2^3\}\{3\} + \{2^2\}\{3\} - \{1^4\}\{3\} + \{3^2\,1\}\{-1\}$

$+ \{3\,2\,1^2\}\{-1\} + \{2\,1^3\}\{-1\} + \{1^2\}\{-1\} + \{3\,2\}\{-5\}$

$+ \{2^2\,1\}\{-5\} + \{2\,1^3\}\{-5\} + \{0\}\{-5\} + \{2\,1\}\{-9\}$

$+ \{1^3\}\{-9\} + \{1\}\{-13\}$

$[S ; 1^5]+$

$+ \{0\}\{15\} + \{1^3\}\{11\} + \{2^3\}\{7\} + \{1\}\{7\} + \{3^3\}\{3\}$

$+ \{2\,1^2\}\{3\} + \{3\,2^2\}\{-1\} + \{2\}\{-1\} + \{3\,1^2\}\{-5\}$

$+ \{3\}\{-9\}$

$[S ; 1^5]-$

$+ \{3^4\}\{9\} + \{3^2\,2^2\}\{5\} + \{3^2\,1^2\}\{1\} + \{2^4\}\{1\} + \{3^2\}\{-3\}$

$+ \{2^2\,1^2\}\{-3\} + \{2^2\}\{-7\} + \{1^4\}\{-7\} + \{1^2\}\{-11\}$

$+ \{0\}\{-15\}$

$[2]$

$+ \{2\}\{4\} + \{2\,1^3\}\{0\} + \{2^4\}\{-4\}$

$[2\,1]$

$+ \{2\,1\}\{6\} + \{3\,1^3\}\{2\} + \{2^2\,1^2\}\{2\} + \{1\}\{2\}$

$+ \{3\,2^3\}\{-2\} + \{2\,1^2\}\{-2\} + \{1^4\}\{-2\} + \{2\,1^3\}\{-6\}$

$[2\,1^2]$

$+ \{2\,1^2\}\{8\} + \{3\,2\,1^2\}\{4\} + \{2^3\,1\}\{4\} + \{2\}\{4\}$

$+ \{1^2\}\{4\} + \{3^2\,2^2\}\{0\} + \{3\,1^2\}\{0\} + \{2^2\,1\}\{0\}$

$+ 2\{2\,1^3\}\{0\} + \{3\,2^2\,1\}\{-4\} + \{2^4\}\{-4\} + \{2\,1\}\{-4\}$

$+ \{1^3\}\{-4\} + \{2^2\,1^2\}\{-8\}$

$[2\,1^3]$

$+ \{2\,1^3\}\{10\} + \{3\,2^2\,1\}\{6\} + \{2^4\}\{6\} + \{2\,1\}\{6\}$

$+ \{1^3\}\{6\} + \{3^3\,2\}\{2\} + \{3\,2\,1\}\{2\} + \{3\,1^3\}\{2\}$

$+ \{2^3\}\{2\} + 2\{2^2\,1^2\}\{2\} + \{1\}\{2\} + \{3^2\,2\,1\}\{-2\}$

$+ \{3\,2^3\}\{-2\} + \{3\,1\}\{-2\} + \{2^2\}\{-2\} + 2\{2\,1^2\}\{-2\}$

$+ \{1^4\}\{-2\} + \{3\,2\,1^2\}\{-6\} + \{2^3\,1\}\{-6\} + \{2\}\{-6\}$

$+ \{1^2\}\{-6\} + \{2\,1^3\}\{-10\}$

$[2\,1^4]+$

$+ \{1\}\{22\} + \{2\,1^2\}\{18\} + \{1^4\}\{18\} + \{3\,2^2\}\{14\}$

$+ \{2^3\,1\}\{14\} + \{2\}\{14\} + \{1^2\}\{14\} + \{3^3\,1\}\{10\}$

$+ \{3\,1^3\}\{10\} + \{2^2\,1\}\{10\} + \{2\,1^3\}\{10\} + \{3\,2^2\,1\}\{6\}$

$+ \{3\}\{6\} + \{2\,1\}\{6\} + \{3\,1^3\}\{2\}$

$[2\,1^4]-$

$+ \{3\,2^3\}\{18\} + \{3^4\}\{14\} + \{3\,2\,1^2\}\{14\} + \{2\,1^3\}\{14\}$

$+ \{3^2\,2^2\}\{10\} + \{3\,2\}\{10\} + \{2^2\,1\}\{10\} + \{2\,1^3\}\{10\}$

$+ \{2^2\,1^2\}\{6\} + \{2^4\}\{6\} + \{2\,1\}\{6\} + \{1^3\}\{6\}$

$+ \{2^2\,1^2\}\{2\} + \{1\}\{2\} + \{1^4\}\{-2\}$

$[2^2]$

$+ \{2^2\}\{8\} + \{3\,2\,1^2\}\{4\} + \{1^2\}\{4\} + \{4\,2^3\}\{0\}$
$+ \{2^2\,1\}\{0\} + \{2\,1^3\}\{0\} + \{0\}\{0\} + \{3\,2^2\,1\}\{-4\}$
$+ \{1^3\}\{-4\} + \{2^3\}\{-8\}$

$[2^2\,1]$

$+ \{2^2\,1\}\{10\} + \{3^2\,1^2\}\{6\} + \{3\,2^2\,1\}\{6\} + \{2\,1\}\{6\}$
$+ \{1^3\}\{6\} + \{4\,3\,2^2\}\{2\} + \{3\,2\,1\}\{2\} + \{3\,1^3\}\{2\}$
$+ \{2^3\}\{2\} + 2\{2^2\,1^2\}\{2\} + \{1\}\{2\} + \{4\,2^2\,1\}\{-2\}$
$+ \{3^2\,2\,1\}\{-2\} + \{3\,2^3\}\{-2\} + \{2^2\}\{-2\} + 2\{2^2\,1\}\{-2\}$
$+ \{1^4\}\{-2\} + \{3\,2^2\}\{-6\} + \{3\,2\,1^2\}\{-6\} + \{2^3\,1\}\{-6\}$
$+ \{1^2\}\{-6\} + \{2^2\,1\}\{-10\}$

$[2^2\,1^2]$

$+ \{2^2\,1^2\}\{12\} + \{3^2\,2\,1\}\{8\} + \{3\,2^3\}\{8\} + \{2^2\}\{8\}$
$+ \{2\,1^2\}\{8\} + \{1^4\}\{8\} + \{4\,3^2\,2\}\{4\} + \{3^2\,1\}\{4\}$
$+ \{3\,2^2\}\{4\} + 2\{3\,2\,1^2\}\{4\} + 2\{2^3\,1\}\{4\} + 2\{1^2\}\{4\}$
$+ \{4\,3\,2\,1\}\{0\} + \{4\,2^3\}\{0\} + \{3^3\,1\}\{0\} + \{3^2\,2^2\}\{0\}$
$+ \{3\,2\}\{0\} + \{3\,1^2\}\{0\} + 3\{2^2\,1\}\{0\} + 2\{2\,1^3\}\{0\}$
$+ \{0\}\{0\} + \{4\,2\,1^2\}\{-4\} + \{3\,2^2\}\{-4\} + \{3^2\,1^2\}\{-4\}$
$+ 2\{3\,2^2\,1\}\{-4\} + 2\{2\,1\}\{-4\} + 2\{1^3\}\{-4\} + \{3\,2\,1\}\{-8\}$
$+ \{3\,1^3\}\{-8\} + \{2^3\}\{-8\} + \{2^2\,1^2\}\{-8\} + \{1\}\{-8\}$
$+ \{2\,1^2\}\{-12\}$

$[2^2\,1^3]+$

$+ \{1^2\}\{24\} + \{2^2\,1\}\{20\} + \{2\,1^3\}\{20\} + \{0\}\{20\}$
$+ \{3^2\,2\}\{16\} + \{3\,2^2\,1\}\{16\} + \{2\,1\}\{16\} + 2\{1^3\}\{16\}$
$+ \{4\,3^2\,1\}\{12\} + \{3\,2\,1\}\{12\} + \{3\,1^3\}\{12\} + 2\{2^3\}\{12\}$
$+ \{2^2\,1^2\}\{12\} + \{1\}\{12\} + \{4\,2^2\,1\}\{8\} + \{3^3\}\{8\}$
$+ \{3^2\,2\,1\}\{8\} + \{3\,1\}\{8\} + 2\{2\,1^2\}\{8\} + \{4\,1^3\}\{4\}$
$+ \{3\,2^2\}\{4\} + \{3\,2\,1^2\}\{4\} + \{2\}\{4\} + \{3\,1^2\}\{0\}$

$[2^2\,1^3]-$

$+ \{3^2\,2^2\}\{20\} + \{4\,3^3\}\{16\} + \{3^2\,1^2\}\{16\} + \{3\,2^2\,1\}\{16\}$
$+ \{2^4\}\{16\} + \{4\,3\,2^2\}\{12\} + \{3^3\,2\}\{12\} + \{3^2\}\{12\}$
$+ \{3\,2\,1\}\{12\} + 2\{2^2\,1^2\}\{12\} + \{4\,3\,1^2\}\{8\} + \{3^2\,2\,1\}\{8$
$+ \{3\,2^2\}\{8\} + 2\{2^2\}\{8\} + \{2\,1^2\}\{8\} + \{1^4\}\{8\}$
$+ \{3^2\,1\}\{4\} + \{3\,2\,1^2\}\{4\} + \{2^3\,1\}\{4\} + 2\{1^2\}\{4\}$
$+ \{2^2\,1\}\{0\} + \{2\,1^3\}\{0\} + \{0\}\{0\} + \{1^3\}\{-4\}$

$[2^3]$

$+ \{2^3\}\{12\} + \{3^2\,2\,1\}\{8\} + \{2\,1^2\}\{8\} + \{4^2\,2^2\}\{4\}$
$+ \{3\,2^2\}\{4\} + \{3\,2\,1^2\}\{4\} + \{2^3\,1\}\{4\} + \{2\}\{4\}$
$+ \{4\,3\,2\,1\}\{0\} + \{3^2\,2^2\}\{0\} + \{3\,1^2\}\{0\} + \{2^2\,1\}\{0\}$
$+ \{2\,1^3\}\{0\} + \{4\,2^2\}\{-4\} + \{3^2\,1^2\}\{-4\} + \{3\,2^2\,1\}\{-4\}$
$+ \{2^4\}\{-4\} + \{2\,1\}\{-4\} + \{3\,2\,1\}\{-8\} + \{2^2\,1^2\}\{-8\}$
$+ \{2^2\}\{-12\}$

$[2^3\,1]$

$+ \{2^3\,1\}\{14\} + \{3^3\,1\}\{10\} + \{3^2\,2^2\}\{10\} + \{2^2\,1\}\{10\}$
$+ \{2\,1^3\}\{10\} + \{4^2\,3\,2\}\{6\} + \{3^2\,2\}\{6\} + \{3^2\,1^2\}\{6\}$
$+ 2\{3\,2^2\,1\}\{6\} + \{2^4\}\{6\} + \{2\,1\}\{6\} + \{1^3\}\{6\}$
$+ \{4^2\,2\,1\}\{2\} + \{4\,3^2\,1\}\{2\} + \{4\,3\,2^2\}\{2\} + \{3^3\,2\}\{2\}$
$+ 2\{3\,2\,1\}\{2\} + \{3\,1^3\}\{2\} + \{2^3\}\{2\} + 2\{2^2\,1^2\}\{2\}$

$+ \{1\}\{2\} + \{4\ 3\ 2\}\{-2\} + \{4\ 3\ 1^2\}\{-2\} + \{4\ 2^2\ 1\}\{-2\}$
$+ 2\{3^2\ 2\ 1\}\{-2\} + \{3\ 2^3\}\{-2\} + \{3\ 1\}\{-2\} + \{2^2\}\{-2\}$
$+ 2\{2\ 1^2\}\{-2\} + \{1^4\}\{-2\} + \{4\ 2\ 1\}\{-6\} + \{3^2\ 1\}\{-6\}$
$+ \{3\ 2^2\}\{-6\} + 2\{3\ 2\ 1^2\}\{-6\} + \{2^3\ 1\}\{-6\} + \{2\}\{-6\}$
$+ \{1^2\}\{-6\} + \{3\ 2\}\{-10\} + \{3\ 1^2\}\{-10\} + \{2^2\ 1\}\{-10\}$
$+ \{2\ 1^3\}\{-10\} + \{2\ 1\}\{-14\}$

$[2^3\ 1^2]+$

$+ \{1^3\}\{26\} + \{2^3\}\{22\} + \{2^2\ 1^2\}\{22\} + \{1\}\{22\}$
$+ \{3^3\}\{18\} + \{3^2\ 2\ 1\}\{18\} + 2\{2\ 1^2\}\{18\} + \{1^4\}\{18\}$
$+ \{4^2\ 3\ 1\}\{14\} + 2\{3\ 2^2\}\{14\} + \{3\ 2\ 1^2\}\{14\} + \{2^3\ 1\}\{14\}$
$+ \{2\}\{14\} + \{1^2\}\{14\} + \{4\ 3^2\}\{10\} + \{4\ 3\ 2\ 1\}\{10\}$
$+ \{3^3\ 1\}\{10\} + 2\{3\ 1^2\}\{10\} + \{2^2\ 1\}\{10\} + \{2\ 1^3\}\{10\}$
$+ \{4\ 2^2\}\{6\} + \{4\ 2\ 1^2\}\{6\} + \{3^2\ 2\}\{6\} + \{3\ 2^2\ 1\}\{6\}$
$+ \{3\}\{6\} + \{2\ 1\}\{6\} + \{4\ 1^2\}\{2\} + \{3\ 2\ 1\}\{2\}$
$+ \{3\ 1^3\}\{2\} + \{3\ 1\}\{-2\}$

$[2^3\ 1^2]-$

$+ \{3^3\ 2\}\{22\} + \{4^2\ 3^2\}\{18\} + \{3^2\ 2\ 1\}\{18\} + \{3\ 2^3\}\{18\}$
$+ \{4^2\ 2^2\}\{14\} + \{4\ 3^2\ 2\}\{14\} + \{3^4\}\{14\} + \{3^2\ 1\}\{14\}$
$+ \{3\ 2\ 1^2\}\{14\} + \{2^3\ 1\}\{14\} + \{4^2\ 1^2\}\{10\} + \{4\ 3\ 2\ 1\}\{10\}$
$+ 2\{3^2\ 2^2\}\{10\} + \{3\ 2\}\{10\} + \{2^2\ 1\}\{10\} + \{2\ 1^3\}\{10\}$
$+ \{4\ 3\ 1\}\{6\} + 2\{3^2\ 1^2\}\{6\} + \{3\ 2^2\ 1\}\{6\} + \{2^4\}\{6\}$
$+ \{2\ 1\}\{6\} + \{1^3\}\{6\} + \{3^2\}\{2\} + \{3\ 2\ 1\}\{2\}$
$+ 2\{2^2\ 1^2\}\{2\} + \{1\}\{2\} + \{2^2\}\{-2\} + \{2\ 1^2\}\{-2\}$
$+ \{1^4\}\{-2\} + \{1^2\}\{-6\}$

$[2^4]$

$+ \{2^4\}\{16\} + \{3^3\ 2\}\{12\} + \{2^2\ 1^2\}\{12\} + \{4^3\ 2\}\{8\}$
$+ \{3^2\ 2\ 1\}\{8\} + \{3\ 2^3\}\{8\} + \{2^2\}\{8\} + \{1^4\}\{8\}$
$+ \{4^2\ 3\ 1\}\{4\} + \{4\ 3^2\ 2\}\{4\} + \{3^2\ 1\}\{4\} + \{3\ 2\ 1\}\{4\}$
$+ \{2^3\ 1\}\{4\} + \{1^2\}\{4\} + \{4^2\ 2\}\{0\} + \{4\ 3\ 2\ 1\}\{0\}$
$+ \{4\ 2^3\}\{0\} + \{3^3\ 1\}\{0\} + \{3\ 2\}\{0\} + \{2^2\ 1\}\{0\}$
$+ \{2\ 1^3\}\{0\} + \{0\}\{0\} + \{4\ 3\ 1\}\{-4\} + \{4\ 2\ 1^2\}\{-4\}$
$+ \{3^2\ 2\}\{-4\} + \{3\ 2^2\ 1\}\{-4\} + \{2\ 1\}\{-4\} + \{1^3\}\{-4\}$
$+ \{4\ 2\}\{-8\} + \{3\ 2\ 1\}\{-8\} + \{3\ 1^3\}\{-8\} + \{2^3\}\{-8\}$
$+ \{1\}\{-8\} + \{3\ 1\}\{-12\} + \{2\ 1^2\}\{-12\} + \{2\}\{-16\}$

$[2^4\ 1]+$

$+ \{1^4\}\{28\} + \{2^3\ 1\}\{24\} + \{1^2\}\{24\} + \{3^3\ 1\}\{20\}$
$+ \{2^2\ 1\}\{20\} + \{2\ 1^3\}\{20\} + \{0\}\{20\} + \{4^3\ 1\}\{16\}$
$+ \{3^2\ 2\}\{16\} + \{3\ 2^2\ 1\}\{16\} + \{2\ 1\}\{16\} + \{1^3\}\{16\}$
$+ \{4^2\ 3\}\{12\} + \{4\ 3^2\ 1\}\{12\} + \{3\ 2\ 1\}\{12\} + \{2\ 1^3\}\{12\}$
$+ \{2^3\}\{12\} + \{1\}\{12\} + \{4\ 3\ 2\}\{8\} + \{4\ 2^2\ 1\}\{8\}$
$+ \{3^3\}\{8\} + \{3\ 1\}\{8\} + \{2\ 1^2\}\{8\} + \{4\ 2\ 1\}\{4\}$
$+ \{4\ 1^3\}\{4\} + \{3\ 2^2\}\{4\} + \{2\}\{4\} + \{4\ 1\}\{0\}$
$+ \{3\ 1^2\}\{0\} + \{3\}\{-4\}$

$[2^4\ 1]-$

$+ \{3^4\}\{24\} + \{4^3\ 3\}\{20\} + \{3^2\ 2^2\}\{20\} + \{4^2\ 3\ 2\}\{16\}$
$+ \{4\ 3^3\}\{16\} + \{3^2\ 1^2\}\{16\} + \{2^4\}\{16\} + \{4^2\ 2\ 1\}\{12\}$
$+ \{4\ 3\ 2^2\}\{12\} + \{3^3\ 2\}\{12\} + \{3^2\}\{12\} + \{2^2\ 1^2\}\{12\}$
$+ \{4^2\ 1\}\{8\} + \{4\ 3\ 1^2\}\{8\} + \{3^2\ 2\ 1\}\{8\} + \{3\ 2^3\}\{8\}$
$+ \{2^2\}\{8\} + \{1^4\}\{8\} + \{4\ 3\}\{4\} + \{3^2\ 1\}\{4\}$

$+ \{3\ 2\ 1^2\}\{4\} + \{2^3\ 1\}\{4\} + \{1^2\}\{4\} + \{3\ 2\}\{0\}$

$+ \{2^2\ 1\}\{0\} + \{2^3\ 1\}\{0\} + \{0\}\{0\} + \{2\ 1\}\{-4\}$

$+ \{1^3\}\{-4\} + \{1\}\{-8\}$

$[2^5]+$

$+ \{0\}\{30\} + \{1^3\}\{26\} + \{2^3\}\{22\} + \{1\}\{22\} + \{3^3\}\{18\}$

$+ \{2\ 1^2\}\{18\} + \{4^3\}\{14\} + \{3\ 2^2\}\{14\} + \{2\}\{14\}$

$+ \{4\ 3^2\}\{10\} + \{3\ 1^2\}\{10\} + \{4\ 2^2\}\{6\} + \{3\}\{6\}$

$+ \{4\ 1^2\}\{2\} + \{4\}\{-2\}$

$[2^5]-$

$+ \{4^4\}\{22\} + \{4^2\ 3^2\}\{18\} + \{4^2\ 2^2\}\{14\} + \{3^4\}\{14\}$

$+ \{4^2\ 1^2\}\{10\} + \{3^2\ 2^2\}\{10\} + \{4^2\}\{6\} + \{3^2\ 1^2\}\{6\}$

$+ \{2^4\}\{6\} + \{3^2\}\{2\} + \{2^2\ 1^2\}\{2\} + \{2^2\}\{-2\} + \{1^4\}\{-2\}$

$+ \{1^2\}\{-6\} + \{0\}\{-10\}$

APPENDIX III: Branching rules for $SU_5 \downarrow SU_2 \times SU_3 \times U_1$

These rules have been evaluated using equation (4.4). The U_1 factors are six times those given by King (1981b) to ensure integer values.

{1}

$+ \{0\}\{1\}\{2\} + \{1\}\{0\}\{-3\}$

{11}

$+ \{0\}\{1^2\}\{4\} + \{1\}\{1\}\{-1\} + \{0\}\{0\}\{-6\}$

{2}

$+ \{0\}\{2\}\{4\} + \{1\}\{1\}\{-1\} + \{2\}\{0\}\{-6\}$

{111}

$+ \{0\}\{0\}\{6\} + \{1\}\{1^2\}\{1\} + \{0\}\{1\}\{-4\}$

{21}

$+ \{0\}\{2\,1\}\{6\} + \{1\}\{2\}\{1\} + \{1\}\{1^2\}\{1\}$

$+ \{2\}\{1\}\{-4\} + \{0\}\{1\}\{-4\} + \{1\}\{0\}\{-9\}$

{3}

$+ \{0\}\{3\}\{6\} + \{1\}\{2\}\{1\} + \{2\}\{1\}\{-4\}$

$+ \{3\}\{0\}\{-9\}$

{1111}

$+ \{1\}\{0\}\{3\} + \{0\}\{1^2\}\{-2\}$

{211}

$+ \{0\}\{1\}\{8\} + \{1\}\{2\,1\}\{3\} + \{0\}\{2\}\{-2\}$

$+ \{1\}\{0\}\{3\} + \{2\}\{1^2\}\{-2\} + \{0\}\{1^2\}\{-2\}$

$+ \{1\}\{1\}\{-7\}$

{22}

$+ \{0\}\{2^2\}\{8\} + \{1\}\{2\,1\}\{3\} + \{2\}\{2\}\{-2\}$

$+ \{0\}\{1^2\}\{-2\} + \{1\}\{1\}\{-7\} + \{0\}\{0\}\{-12\}$

{31}

$+ \{0\}\{3\,1\}\{8\} + \{1\}\{3\}\{3\} + \{1\}\{2\,1\}\{3\}$

$+ \{2\}\{2\}\{-2\} + \{0\}\{2\}\{-2\} + \{2\}\{1^2\}\{-2\}$

$+ \{3\}\{1\}\{-7\} + \{1\}\{1\}\{-7\} + \{2\}\{0\}\{-12\}$

{4}

$+ \{0\}\{4\}\{8\} + \{1\}\{3\}\{3\} + \{2\}\{2\}\{-2\}$

$+ \{3\}\{1\}\{-7\} + \{4\}\{0\}\{-12\}$

{11111}

$+ \{0\}\{0\}\{0\}$

{2111}

$+ \{1\}\{1\}\{5\} + \{0\}\{2\,1\}\{0\} + \{2\}\{0\}\{0\}$

$+ \{0\}\{0\}\{0\} + \{1\}\{1^2\}\{-5\}$

{221}

$+ \{0\}\{1^2\}\{10\} + \{1\}\{2^2\}\{5\} + \{1\}\{1\}\{5\}$

$+ \{2\}\{2\,1\}\{0\} + \{0\}\{2\,1\}\{0\} + \{1\}\{2\}\{-5\}$

$+ \{0\}\{0\}\{0\} + \{1\}\{1^2\}\{-5\} + \{0\}\{1\}\{-10\}$

{311}

$+ \{0\}\{2\}\{10\} + \{1\}\{3\,1\}\{5\} + \{0\}\{3\}\{0\}$

$+ \{1\}\{1\}\{5\} + \{2\}\{2\,1\}\{0\} + \{0\}\{2\,1\}\{0\}$

$+ \{1\}\{2\}\{-5\} + \{2\}\{0\}\{0\} + \{3\}\{1^2\}\{-5\}$

$+ \{1\}\{1^2\}\{-5\} + \{2\}\{1\}\{-10\}$

{32}

$+ \{0\}\{3\ 2\}\{10\} + \{1\}\{3\ 1\}\{5\} + \{2\}\{3\}\{0\}$

$+ \{1\}\{2^2\}\{5\} + \{2\}\{2\ 1\}\{0\} + \{0\}\{2\ 1\}\{0\}$

$+ \{3\}\{2\}\{-5\} + \{1\}\{2\}\{-5\} + \{1\}\{1^2\}\{-5\}$

$+ \{2\}\{1\}\{-10\} + \{0\}\{1\}\{-10\} + \{1\}\{0\}\{-15\}$

{41}

$+ \{0\}\{4\ 1\}\{10\} + \{1\}\{4\}\{5\} + \{1\}\{3\ 1\}\{5\}$

$+ \{2\}\{3\}\{0\} + \{0\}\{3\}\{0\} + \{2\}\{2\ 1\}\{0\}$

$+ \{3\}\{2\}\{-5\} + \{1\}\{2\}\{-5\} + \{3\}\{1^2\}\{-5\}$

$+ \{4\}\{1\}\{-10\} + \{2\}\{1\}\{-10\} + \{3\}\{0\}\{-15\}$

{5}

$+ \{0\}\{5\}\{10\} + \{1\}\{4\}\{5\} + \{2\}\{3\}\{0\}$

$+ \{3\}\{2\}\{-5\} + \{4\}\{1\}\{-10\} + \{5\}\{0\}\{-15\}$

{21111}

$+ \{0\}\{1\}\{2\} + \{1\}\{0\}\{-3\}$

{2211}

$+ \{1\}\{1^2\}\{7\} + \{0\}\{2^2\}\{2\} + \{2\}\{1\}\{2\}$

$+ \{0\}\{1\}\{2\} + \{1\}\{2\ 1\}\{-3\} + \{1\}\{0\}\{-3\}$

$+ \{0\}\{1^2\}\{-8\}$

{222}

$+ \{0\}\{0\}\{12\} + \{1\}\{1^2\}\{7\} + \{2\}\{2^2\}\{2\}$

$+ \{0\}\{1\}\{2\} + \{1\}\{2\ 1\}\{-3\} + \{0\}\{2\}\{-8\}$

{3111}

$+ \{1\}\{2\}\{7\} + \{0\}\{3\ 1\}\{2\} + \{2\}\{1\}\{2\}$

$+ \{0\}\{1\}\{2\} + \{1\}\{2\ 1\}\{-3\} + \{3\}\{0\}\{-3\}$

$+ \{1\}\{0\}\{-3\} + \{2\}\{1^2\}\{-8\}$

{321}

$+ \{0\}\{2\ 1\}\{12\} + \{1\}\{3\ 2\}\{7\} + \{1\}\{2\}\{7\}$

$+ \{2\}\{3\ 1\}\{2\} + \{0\}\{3\ 1\}\{2\} + \{1\}\{3\}\{-3\}$

$+ \{1\}\{1^2\}\{7\} + \{2\}\{2^2\}\{2\} + \{0\}\{2^2\}\{2\}$

$+ \{2\}\{1\}\{2\} + \{0\}\{1\}\{2\} + \{3\}\{2\ 1\}\{-3\}$

$+ 2\{1\}\{2\ 1\}\{-3\} + \{2\}\{2\}\{-8\} + \{0\}\{2\}\{-8\}$

$+ \{1\}\{0\}\{-3\} + \{2\}\{1^2\}\{-8\} + \{0\}\{1^2\}\{-8\}$

$+ \{1\}\{1\}\{-13\}$

{33}

$+ \{0\}\{3^2\}\{12\} + \{1\}\{3\ 2\}\{7\} + \{2\}\{3\ 1\}\{2\}$

$+ \{3\}\{3\}\{-3\} + \{0\}\{2^2\}\{2\} + \{1\}\{2\ 1\}\{-3\}$

$+ \{2\}\{2\}\{-8\} + \{0\}\{1^2\}\{-8\} + \{1\}\{1\}\{-13\}$

$+ \{0\}\{0\}\{-18\}$

{411}

$+ \{0\}\{3\}\{12\} + \{1\}\{4\ 1\}\{7\} + \{0\}\{4\}\{2\}$

$+ \{1\}\{2\}\{7\} + \{2\}\{3\ 1\}\{2\} + \{0\}\{3\ 1\}\{2\}$

$+ \{1\}\{3\}\{-3\} + \{2\}\{1\}\{2\} + \{3\}\{2\ 1\}\{-3\}$

$+ \{1\}\{2\ 1\}\{-3\} + \{2\}\{2\}\{-8\} + \{3\}\{0\}\{-3\}$

$+ \{4\}\{1^2\}\{-8\} + \{2\}\{1^2\}\{-8\} + \{3\}\{1\}\{-13\}$

{51}

$+ \{0\}\{5\ 1\}\{12\} + \{1\}\{5\}\{7\} + \{1\}\{4\ 1\}\{7\}$

$+ \{2\}\{4\}\{2\} + \{0\}\{4\}\{2\} + \{2\}\{3\ 1\}\{2\}$

$+ \{3\}\{3\}\{-3\} + \{1\}\{3\}\{-3\} + \{3\}\{2\ 1\}\{-3\}$

$+ \{4\}\{2\}\{-8\} + \{2\}\{2\}\{-8\} + \{4\}\{1^2\}\{-8\}$

$+ \{5\}\{1\}\{-13\} + \{3\}\{1\}\{-13\} + \{4\}\{0\}\{-18\}$

{6}

$+ \{0\}\{6\}\{12\} + \{1\}\{5\}\{7\} + \{2\}\{4\}\{2\}$

$+ \{3\}\{3\}\{-3\} + \{4\}\{2\}\{-8\} + \{5\}\{1\}\{-13\}$

$+ \{6\}\{0\}\{-18\}$

{22111}

$+ \{0\}\{1^2\}\{4\} + \{1\}\{1\}\{-1\} + \{0\}\{0\}\{-6\}$

{2221}

$+ \{1\}\{0\}\{9\} + \{2\}\{1^2\}\{4\} + \{0\}\{1^2\}\{4\}$

$+ \{1\}\{2^2\}\{-1\} + \{1\}\{1\}\{-1\} + \{0\}\{2\ 1\}\{-6\}$

{31111}

$+ \{0\}\{2\}\{4\} + \{1\}\{1\}\{-1\} + \{2\}\{0\}\{-6\}$

{3211}

$+ \{1\}\{2\ 1\}\{9\} + \{0\}\{3\ 2\}\{4\} + \{2\}\{2\}\{4\}$

$+ \{0\}\{2\}\{4\} + \{1\}\{3\ 1\}\{-1\} + \{2\}\{1^2\}\{4\}$

$+ \{0\}\{1^2\}\{4\} + \{1\}\{2^2\}\{-1\} + \{3\}\{1\}\{-1\}$

$+ 2\{1\}\{1\}\{-1\} + \{2\}\{2\ 1\}\{-6\} + \{0\}\{2\ 1\}\{-6\}$

$+ \{2\}\{0\}\{-6\} + \{0\}\{0\}\{-6\} + \{1\}\{1^2\}\{-11\}$

{322}

$+ \{0\}\{1\}\{14\} + \{1\}\{2\ 1\}\{9\} + \{2\}\{3\ 2\}\{4\}$

$+ \{0\}\{2\}\{4\} + \{1\}\{3\ 1\}\{-1\} + \{0\}\{3\}\{-6\}$

$+ \{1\}\{0\}\{9\} + \{2\}\{1^2\}\{4\} + \{0\}\{1^2\}\{4\}$

$+ \{3\}\{2^2\}\{-1\} + \{1\}\{2^2\}\{-1\} + \{1\}\{1\}\{-1\}$

$+ \{2\}\{2\ 1\}\{-6\} + \{0\}\{2\ 1\}\{-6\} + \{1\}\{2\}\{-11\}$

{331}

$+ \{0\}\{2^2\}\{14\} + \{1\}\{3^2\}\{9\} + \{1\}\{2\ 1\}\{9\}$

$+ \{2\}\{3\ 2\}\{4\} + \{0\}\{3\ 2\}\{4\} + \{2\}\{2\}\{4\}$

$+ \{3\}\{3\ 1\}\{-1\} + \{1\}\{3\ 1\}\{-1\} + \{2\}\{3\}\{-6\}$

$+ \{0\}\{1^2\}\{4\} + \{1\}\{2^2\}\{-1\} + \{1\}\{1\}\{-1\}$

$+ \{2\}\{2\ 1\}\{-6\} + \{0\}\{2\ 1\}\{-6\} + \{1\}\{2\}\{-11\}$

$+ \{0\}\{0\}\{-6\} + \{1\}\{1^2\}\{-11\} + \{0\}\{1\}\{-16\}$

{4111}

$+ \{1\}\{3\}\{9\} + \{0\}\{4\ 1\}\{4\} + \{2\}\{2\}\{4\}$

$+ \{0\}\{2\}\{4\} + \{1\}\{3\ 1\}\{-1\} + \{3\}\{1\}\{-1\}$

$+ \{1\}\{1\}\{-1\} + \{2\}\{2\ 1\}\{-6\} + \{4\}\{0\}\{-6\}$

$+ \{2\}\{0\}\{-6\} + \{3\}\{1^2\}\{-11\}$

{421}

$+ \{0\}\{3\ 1\}\{14\} + \{1\}\{4\ 2\}\{9\} + \{1\}\{3\}\{9\}$

$+ \{2\}\{4\ 1\}\{4\} + \{0\}\{4\ 1\}\{4\} + \{1\}\{4\}\{-1\}$

$+ \{1\}\{2\ 1\}\{9\} + \{2\}\{3\ 2\}\{4\} + \{0\}\{3\ 2\}\{4\}$

$+ \{2\}\{2\}\{4\} + \{0\}\{2\}\{4\} + \{3\}\{3\ 1\}\{-1\}$

$+ 2\{1\}\{3\ 1\}\{-1\} + \{2\}\{3\}\{-6\} + \{0\}\{3\}\{-6\}$

$+ \{2\}\{1^2\}\{4\} + \{3\}\{2^2\}\{-1\} + \{1\}\{2^2\}\{-1\}$

$+ \{3\}\{1\}\{-1\} + \{1\}\{1\}\{-1\} + \{4\}\{2\ 1\}\{-6\}$

$+ 2\{2\}\{2\ 1\}\{-6\} + \{0\}\{2\ 1\}\{-6\} + \{3\}\{2\}\{-11\}$

$+ \{1\}\{2\}\{-11\} + \{2\}\{0\}\{-6\} + \{3\}\{1^2\}\{-11\}$

$+ \{1\}\{1^2\}\{-11\} + \{2\}\{1\}\{-16\}$

{43}

+ { 0 }{ 4 3 }{ 14 } + { 1 }{ 4 2 }{ 9 } + { 2 }{ 4 1 }{ 4 }

+ { 3 }{ 4 }{ -1 } + { 1 }{ 3^2 }{ 9 } + { 2 }{ 3 2 }{ 4 }

+ { 0 }{ 3 2 }{ 4 } + { 3 }{ 3 1 }{ -1 } + { 1 }{ 3 1 }{ -1 }

+ { 4 }{ 3 }{ -6 } + { 2 }{ 3 }{ -6 } + { 1 }{ 2^2 }{ -1 }

+ { 2 }{ 2 1 }{ -6 } + { 0 }{ 2 1 }{ -6 } + { 3 }{ 2 }{ -11 }

+ { 1 }{ 2 }{ -11 } + { 1 }{ 1^2 }{ -11 } + { 2 }{ 1 }{ -16 }

+ { 0 }{ 1 }{ -16 } + { 1 }{ 0 }{ -21 }

{511}

+ { 0 }{ 4 }{ 14 } + { 1 }{ 5 1 }{ 9 } + { 0 }{ 5 }{ 4 }

+ { 1 }{ 3 }{ 9 } + { 2 }{ 4 1 }{ 4 } + { 0 }{ 4 1 }{ 4 }

+ { 1 }{ 4 }{ -1 } + { 2 }{ 2 }{ 4 } + { 3 }{ 3 1 }{ -1 }

+ { 1 }{ 3 1 }{ -1 } + { 2 }{ 3 }{ -6 } + { 3 }{ 1 }{ -1 }

+ { 4 }{ 2 1 }{ -6 } + { 2 }{ 2 1 }{ -6 } + { 3 }{ 2 }{ -11 }

+ { 4 }{ 0 }{ -6 } + { 5 }{ 1^2 }{ -11 } + { 3 }{ 1^2 }{ -11 }

+ { 4 }{ 1 }{ -16 }

{52}

+ { 0 }{ 5 2 }{ 14 } + { 1 }{ 5 1 }{ 9 } + { 2 }{ 5 }{ 4 }

+ { 1 }{ 4 2 }{ 9 } + { 2 }{ 4 1 }{ 4 } + { 0 }{ 4 1 }{ 4 }

+ { 3 }{ 4 }{ -1 } + { 1 }{ 4 }{ -1 } + { 2 }{ 3 2 }{ 4 }

+ { 3 }{ 3 1 }{ -1 } + { 1 }{ 3 1 }{ -1 } + { 4 }{ 3 }{ -6 }

+ { 2 }{ 3 }{ -6 } + { 0 }{ 3 }{ -6 } + { 3 }{ 2^2 }{ -1 }

+ { 4 }{ 2 1 }{ -6 } + { 2 }{ 2 1 }{ -6 } + { 5 }{ 2 }{ -11 }

+ { 3 }{ 2 }{ -11 } + { 1 }{ 2 }{ -11 } + { 3 }{ 1^2 }{ -11 }

+ { 4 }{ 1 }{ -16 } + { 2 }{ 1 }{ -16 } + { 3 }{ 0 }{ -21 }

{61}

+ { 0 }{ 6 1 }{ 14 } + { 1 }{ 6 }{ 9 } + { 1 }{ 5 1 }{ 9 }

+ { 2 }{ 5 }{ 4 } + { 0 }{ 5 }{ 4 } + { 2 }{ 4 1 }{ 4 }

+ { 3 }{ 4 }{ -1 } + { 1 }{ 4 }{ -1 } + { 3 }{ 3 1 }{ -1 }

+ { 4 }{ 3 }{ -6 } + { 2 }{ 3 }{ -6 } + { 4 }{ 2 1 }{ -6 }

+ { 5 }{ 2 }{ -11 } + { 3 }{ 2 }{ -11 } + { 5 }{ 1^2 }{ -11 }

+ { 6 }{ 1 }{ -16 } + { 4 }{ 1 }{ -16 } + { 5 }{ 0 }{ -21 }

{7}

+ { 0 }{ 7 }{ 14 } + { 1 }{ 6 }{ 9 } + { 2 }{ 5 }{ 4 }

+ { 3 }{ 4 }{ -1 } + { 4 }{ 3 }{ -6 } + { 5 }{ 2 }{ -11 }

+ { 6 }{ 1 }{ -16 } + { 7 }{ 0 }{ -21 }

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